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The simple zeros of the Riemann zeta-function

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Abstract

The Simple Zeros of the Riemann Zeta-Function

by Melissa Miller

There have been many tables of primes produced since antiquity. In 348 BC Plato studied the divisors of the number 5040. In 1202 Fibonacci gave an example with a list of prime numbers up to 100. By the 1770's a table of number factorizations up to two million was constructed. In 1859 Riemann demonstrated that the key to the deeper understanding of the distribution of prime numbers lies in the study of a certain complex-valued function, called the zeta-function. In 1973 Montgomery used explicit formulas to study the pair correlation of the zeros of the zeta-function and their relationship to primes. It is conjectured that all the zeros of the zeta-function are simple. Montgomery proved that at least two-thirds of the zeros are simple. In this thesis I provide complete proofs of Montgomery's method and its applications to simple zeros and differences between consecutive primes. In addition, I give a proof of the explicit formula derived by Ledoan and Zaharescu for the pair correlation of vertical shifts of zeros of the zeta-function and derive several consequences that may be useful for further study of the zeros.

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Chapter 1

Introduction

There have been many tables of primes produced since antiquity. In 348 BC Plato studied the divisors of 5040. In 1202 Fibonacci gave an example with a list of primes up to 100. By the 1770's a table of number factorizations up to two million was constructed. Figure 1.1 shows a table with the first 54 primes.

1	**2**	**3**	4	**5**	6	**7**	8	9	10	**11**	12	**13**	14	15	16
17	18	**19**	20	21	22	**23**	24	25	26	27	28	**29**	30	**31**	32
33	34	35	36	**37**	38	39	40	**41**	42	**43**	44	45	46	**47**	48
49	50	51	52	**53**	54	55	56	57	58	**59**	60	**61**	62	63	64
65	66	**67**	68	69	70	**71**	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90	91	92	93	94	95	96
97	98	99	100	**101**	102	**103**	104	105	106	**107**	108	**109**	110	111	112
113	114	115	116	117	118	119	120	121	122	123	124	125	126	**127**	128
129	130	**131**	132	133	134	135	136	**137**	138	**139**	140	141	142	143	144
145	146	147	148	**149**	150	**151**	152	153	154	155	156	**157**	158	159	160
161	162	**163**	164	165	166	**167**	168	169	170	171	172	**173**	174	175	176
177	178	**179**	180	**181**	182	183	184	185	186	187	188	189	190	**191**	192
193	194	195	196	**197**	198	**199**	200	201	202	203	204	205	206	207	208
209	210	**211**	212	213	214	215	216	217	218	219	220	221	222	**223**	224
225	226	**227**	228	**229**	230	231	232	**233**	234	235	236	237	238	**239**	240
241	242	243	244	245	246	247	248	249	250	**251**	252	253	254	255	256

Figure 1.1: The first 54 primes among the positive integers up to 256.

The first mathematician of all time to make any significant conjecture about the distribution of the primes was Legendre. In 1808, he conjectured that, for large values of x, the number of primes not exceeding x, which is denoted by $\pi(x)$, is given approximately by $x/(\log x - 1.08\ldots)$. The tables of primes show that the detailed distribution of the primes is erratic, but the pictures in Figure 1.2 show that the counting function $\pi(x)$ increases steadily.

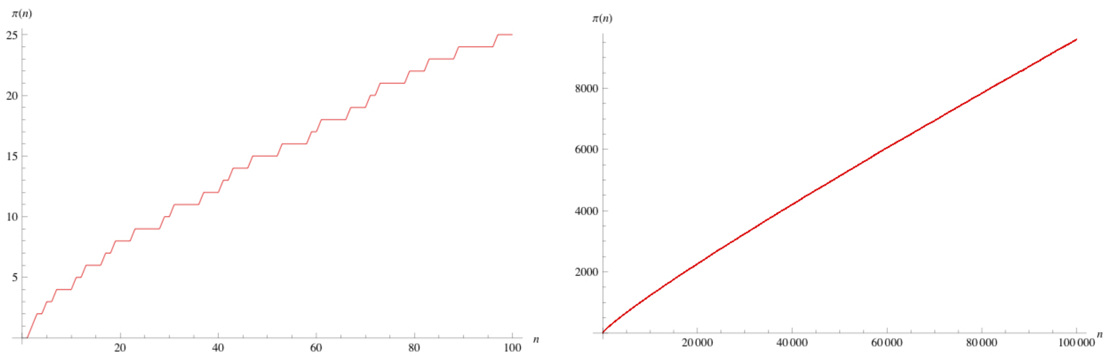

Figure 1.2: The left-hand plot is a picture of $\pi(x)$ for $x \in [0, 100]$. The right-hand plot shows $\pi(x)$ for $x \in [0, 100000]$.

In 1849 Gauss conjectured that $\pi(x) \sim \mathrm{Li}(x)$ as $x \to \infty$, where $\mathrm{Li}(x) = \int_2^x dt/\log t$.[1] His conjecture was proved in 1896 by Hadamard and de la Vallée Poussin, working independently. Their result became known as the Prime Number Theorem, which is that $\pi(x) \sim x/\log x$ as $x \to \infty$ and can be interpreted as the probability that a number n close to x being prime is asymptotically $1/\log x$, and hence $1/\log n$.

A striking event in the early development of analytic number theory was the publication of Riemann's epoch-making discovery [7] in 1859. Building on the work of Cauchy, Riemann demonstrated that the key to the deeper understanding of the distribution of the primes lies in the study of the zeta-function $\zeta(s)$ as a function of the complex variable $s = \sigma + it$. The function $\zeta(s)$ has its origin in Euler's identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.1}$$

which shows the connection that exists between $\zeta(s)$ and primes. The infinite series defines an analytic function $\zeta(s)$, regular for $\mathrm{Re}(s) = \sigma > 1$. The product is also absolutely

[1]For example, we have $\pi(10^{16}) = 279,238,341,033,925$ and $\pi(10^{16})/\mathrm{Li}(10^{16}) = 0.999999989\ldots$.

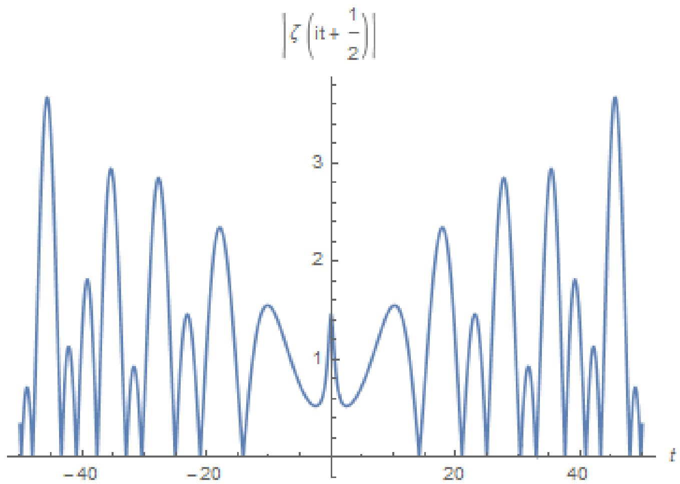

Figure 1.3: Zeros of $\zeta(s)$ on the critical line, with $t \in [-50, 50]$.

convergent for $\sigma > 1$. In his memoir Riemann proved two main results:

(a) The function $\zeta(s)$ can be continued analytically over the whole complex plane, except for a simple pole at $s = 1$ with residue 1.

(b) $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.2}$$

where

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt.$$

The functional equation (2.2) allows the properties of $\zeta(s)$ for $\sigma < 0$ to be inferred from its properties for $\sigma > 1$. Riemann found it convenient to define[2]

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = (s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}+1\right)\zeta(s), \tag{1.3}$$

so that the function $\xi(s)$ is entire and that (2.2) takes the form

$$\xi(s) = \xi(1-s). \tag{1.4}$$

Since the infinite series in (2.1) converges for $\sigma > 1$, $\zeta(s)$ has no zeros in this region. Since $1/\Gamma(z)$ is entire, the function $\Gamma(s/2)$ is non-vanishing, so that $\xi(s)$ also has no zeros for

[2]The second expression in (2.3) is obtained by applying the functional relation $\Gamma(s+1) = s\Gamma(s)$.

$\sigma > 1$. By (2.4), the zeros of $\xi(s)$ are thus confined to the critical strip $0 \le \sigma \le 1$. By (2.3), any zero of $\zeta(s)$ must either be a zero of $\xi(s)$, or a pole of $\Gamma(s/2)$. Hence, the zeros of $\zeta(s)$ must lie in the critical strip, with the exception of the trivial zeros at $s = -2, -4, -6, \ldots$ corresponding to poles of $\Gamma(s/2)$. We may observe, further, that if $\rho = \beta + i\gamma$ is a zero of $\xi(s)$, then by (2.4), so is $1 - \rho$. Since $\overline{\xi(s)} = \xi(\bar{s})$, the complex conjugates $\bar{\rho}$ and $1 - \bar{\rho}$ are also zeros. Hence, the zeros of $\zeta(s)$ are positioned symmetrically with the real line and the critical line $\sigma = 1/2$. Figure 1.3 shows the first ten zeros in the upper half-plane and their complex conjugates with respect to the real axis. Figure 1.4 shows the contour plot of $\zeta(s)$, in which a zero is indicated by the intersection of the real part of $\zeta(s)$ and the imaginary part of $\zeta(s)$. There are seven zeros here. Figure 1.5 is a parametric plot shows the zeros according to the number of windings as t ranges up to height 50.

Riemann conjectured that $\beta = 1/2$, so that $\rho = 1/2 + i\gamma$. The numerical evidence for this conjecture is extremely convincing. The first four zeros in the upper half of the critical strip are $1/2+i14.13472\ldots$, $1/2+i21.02203\ldots$, $1/2+i25.01085\ldots$, and $1/2+i30.42487\ldots$. Figure 1.6 shows the remarkable zeta-landscape with the zeros as spikes on the critical line. With this picture in mind, let us mention the following important conjectures.

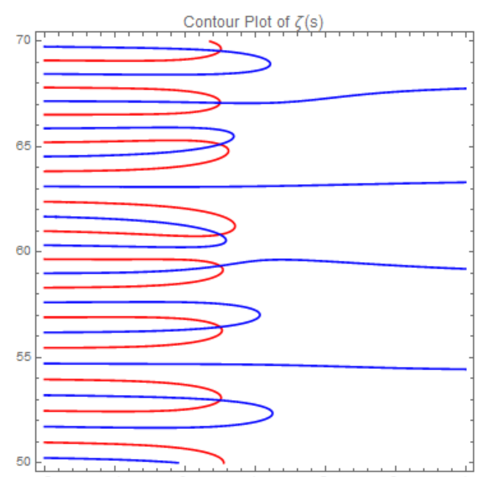

Figure 1.4: Contour plot of $\zeta(s)$.

Riemann Hypothesis. All the complex zeros of $\zeta(s)$ lie on the line $\sigma = 1/2$.

Simple Zeros Conjecture. All the complex zeros of $\zeta(s)$ are simple.

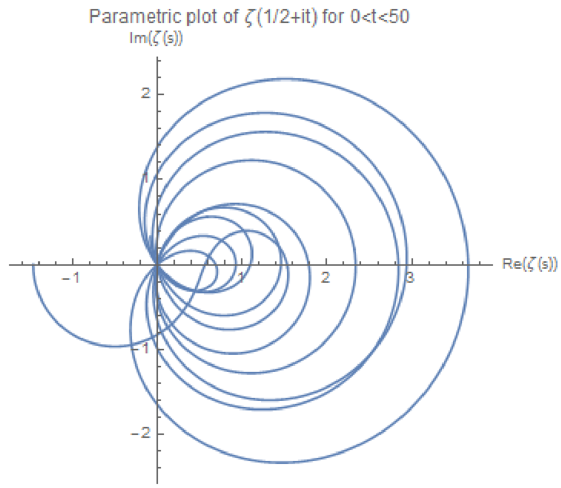

Figure 1.5: Parametric plot of $\zeta(s)$.

On the truth of the Riemann Hypothesis, many of the bounds on prime estimates can be vastly refined and primality proving can be simplified.[3] In 2004, supercomputers with processing capacity at speeds of nanoseconds were used to verify that the first 10 trillion zeros up to height $t = 10^{24}$ are on the critical line and simple.

I shall now describe our project and research objectives. The distribution of the differences of the imaginary parts of pairs of zeros $\mathrm{Im}(\rho - \rho') = \gamma - \gamma'$ was first studied by Montgomery [6] in the early 1970's. He proved on the Riemann Hypothesis that, for any positive ϵ, more than $2/3 - \epsilon$ of the zeros are simple. In 1989 Conrey, Ghosh, and Gonek [2] discovered a new technique that shows (conditionally) that more than $19/17 - \epsilon = 0.70370\ldots$ of the zeros are simple. In 1993, Cheer and Goldston [1] proved (conditionally) that more than $0.67275\ldots$ of the zeros are simple.[4] In 2011 Ledoan and Zaharescu [4], [5] investigated the pair correlation of the vertical shifts of the zeros, that is, the pair correlation of the zeros of $\zeta(s)$ and the zeros of $\zeta(k(s-1/2)+1/2)$ for any fixed positive integer k. Upon reviewing this work for the American Mathematical Society, the Turkish mathematician Cem Yalçın Yıldırım, famous for his recent work on bounded gaps between primes, remarked: "Another

[3]One such consequence is the sharp estimate $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x)$ as $x \to \infty$.

[4]Conrey, Ghosh, and Gonek's lower bound is weaker than Cheer and Goldston's, since their technique depends on a second unproved conjecture on the rate of growth of $\zeta(s)$ on the critical line, called the Lindelöf Hypothesis.

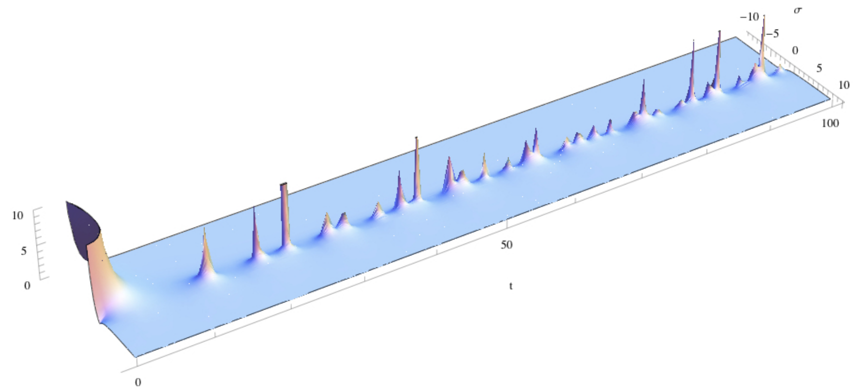

Figure 1.6: Zeros of $\zeta(s)$ on the critical line, with $t \in [0, 100]$, appearing as divergences.

point is that Montgomery, assuming the Riemann Hypothesis, deduced some results about the percentage of simple zeros and about the gaps between the zeros of $\zeta(s)$ from his double sum over the zeta zeros. It would be desirable to have some such conclusions from the authors' formulas." Hence, the aim of our project is to study Montgomery's method and use the explicit formulas for the pair correlation of the vertical shifts of the zeros of $\zeta(s)$ to derive interesting results that may add further information about the simple zeros.

Chapter 2

Montgomery's Method

2.1 Landau-Gonek Explicit Formula

We assume the Riemann Hyposthesis (for short RH). The number $\rho = \frac{1}{2} + i\gamma$ denotes, for each $\gamma \in \mathbb{R}$, a nontrivial zero of the Riemann zeta-function $\zeta(s)$. In 1973 Montgomery [6] investigated the distribution of the differences $\gamma - \Gamma'$ between the zeros, where γ and Γ' run through the imaginary parts of the zeros of $\zeta(s)$. Montgomery manipulated the explicit formula given by the following lemma.

Lemma 1. *(Assuming RH.) For $1 < \sigma < 2$ and $x \geq 1$, we have*

$$
\begin{aligned}
(2\sigma - 1) \sum_{\gamma} & \frac{x^{i\gamma}}{(\sigma - 1/2)^2 + (t-\gamma)^2} \\
&= -x^{-1/2} \left(\sum_{n \leq x} \Lambda(n) \left(\frac{x}{n}\right)^{1-\sigma+it} + \sum_{n > x} \Lambda(n) \left(\frac{x}{n}\right)^{\sigma+it} \right) \\
&\quad + x^{1/2-\sigma+it} (\log \tau + O_\sigma(1)) + O_\sigma(x^{1/2}\tau^{-1}),
\end{aligned} \tag{2.1}
$$

where $\tau = |t| + 2$ and $\Lambda(s)$ denotes the von-Mangoldt function,

$$
\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k,\ p \text{ prime},\ k \geq 1; \\ 0, & \text{otherwise.} \end{cases}
$$

The explicit formula (2.1) provides an explicit link between a weighted count of the primes p^n and a sum over the nontrivial zeros of $\zeta(s)$. It is derived using an explicit formula of Landau, which states that unconditionally for $x > 1$, $x \neq p^n$,

$$\sum_{n \le x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho - s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s}, \tag{2.2}$$

provided that $s \neq 1$, $s \neq \rho$, and $s \neq 2n$.

We proceed to prove Lemma 1. In (2.2), we solve for the sum over all zeros ρ in the critical strip above the real axis to obtain

$$\sum_{\rho} \frac{x^{\rho - s}}{\rho - s} = -\left(\frac{\zeta'(s)}{\zeta(s)} + \sum_{n=x} \frac{\Lambda(n)}{n^s} - \frac{x^{1-s}}{1-s} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s} \right). \tag{2.3}$$

This relation holds independent of RH. However, assuming RH, we have $\rho = \frac{1}{2} + i\gamma$, for each $\gamma \in \mathbb{R}$. Setting $s = \sigma + it$, we obtain $\rho - s = \frac{1}{2} - \sigma + i\gamma - it$. Then (2.3) becomes

$$\sum_{\gamma} \frac{x^{\frac{1}{2} - \sigma + i\gamma - it}}{\frac{1}{2} - \sigma + it - i\gamma} = -\left(\frac{\zeta'(s)}{\zeta(s)} + \sum_{n \le x} \frac{\Lambda(n)}{n^s} - \frac{x^{1-s}}{1-s} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s} \right),$$

from which

$$\sum_{\gamma} \frac{x^{i\gamma - it}}{\sigma - \frac{1}{2} + it - i\gamma} = x^{\sigma - \frac{1}{2}} \left(\frac{\zeta'(s)}{\zeta(s)} + \sum_{n \le x} \frac{\Lambda(n)}{n^s} - \frac{x^{1-s}}{1-s} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s} \right). \tag{2.4}$$

In similar fashion, setting $s = 1 - \sigma + it$, we have $\rho - s = -\frac{1}{2} + \sigma + i\gamma - it$. Then (2.3) becomes

$$\sum_{\gamma} \frac{x^{-\frac{1}{2} + \sigma + i\gamma - it}}{-\frac{1}{2} + \sigma + it - i\gamma} = -\left(\frac{\zeta'(s)}{\zeta(s)} + \sum_{n \le x} \frac{\Lambda(n)}{n^s} - \frac{x^{1-s}}{1-s} - \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s} \right),$$

from which

$$\begin{aligned} &\sum_{\gamma} \frac{x^{i\gamma - it}}{\frac{1}{2} - \sigma + it - i\gamma} \\ &\quad = x^{\frac{1}{2} - \sigma} \left(\frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)} + \sum_{n \le x} \frac{\Lambda(n)}{n^{1-\sigma+it}} - \frac{x^{\sigma - it}}{\sigma - it} - \sum_{n=1}^{\infty} \frac{x^{-2n-1+\sigma-it}}{2n+1-\sigma+it} \right). \end{aligned} \tag{2.5}$$

Subtracting (2.5) from (2.4), we obtain on the one hand

$$\begin{aligned} \sum_{\gamma} \frac{x^{i\gamma - it}}{\sigma - \frac{1}{2} + it - i\gamma} - \sum_{\gamma} \frac{x^{i\gamma - it}}{\frac{1}{2} - \sigma + it - i\gamma} &= \sum_{\gamma} x^{i\gamma - it} \left(\frac{-2\sigma + 1}{-(\sigma - \frac{1}{2})^2 - (t-\gamma)} \right) \\ &= (2\sigma - 1) \sum_{\gamma} \frac{x^{i\gamma - it}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \end{aligned} \tag{2.6}$$

On the other hand, we have

$$
\begin{aligned}
&x^{\sigma-\frac{1}{2}}\left(\frac{\zeta'(s)}{\zeta(s)}+\sum_{n\le x}\frac{\Lambda(n)}{n^s}-\frac{x^{1-s}}{1-s}-\sum_{n=1}^{\infty}\frac{x^{-2n-s}}{2n+s}\right)\\
&\quad-x^{-\frac{1}{2}-\sigma}\left(\frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)}+\sum_{n\le x}\frac{\Lambda(n)}{n^{1-\sigma+it}}-\frac{x^{\sigma-it}}{\sigma-it}-\sum_{n=1}^{\infty}\frac{x^{-2n-1+\sigma-it}}{2n+1-\sigma+it}\right)\\
&\quad=x^{\frac{1}{2}-\sigma}\sum_{n\le x}\frac{\Lambda(n)}{n^{1-\sigma+it}}-x^{\sigma-\frac{1}{2}}\sum_{n>x}\frac{\Lambda(n)}{n^{\sigma+it}}-x^{\frac{1}{2}-\sigma}\frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)}\\
&\qquad+x^{\frac{1}{2}-it}\left(\frac{1}{\sigma-1+it}+\frac{1}{\sigma-it}\right)-x^{-\frac{1}{2}-it}\sum_{n=1}^{\infty}x^{-2n}\left(\frac{1}{2n+\sigma+it}+\frac{1}{2n+1-\sigma+it}\right)\\
&\quad=-x^{-\frac{1}{2}}\left(\sum_{n\le x}\Lambda(n)\left(\frac{x}{n^{it}}\right)^{1-\sigma}+\sum_{n>x}\Lambda(n)\left(\frac{x}{n^{it}}\right)^{\sigma}\right)-x^{\frac{1}{2}-\sigma}\frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)}\\
&\qquad+x^{\frac{1}{2}-it}\left(\frac{2\sigma-1}{(\sigma-1+it)(\sigma-it)}\right)-x^{-\frac{1}{2}-it}\sum_{n=1}^{\infty}\left(\frac{(2\sigma-1)x^{-2n}}{(\sigma-1-2n-it)(\sigma+2n+it)}\right).
\end{aligned}
\tag{2.7}
$$

Equating (2.6) and (2.7) and multiplying both sides of the result by x^{it}, we obtain

$$
\begin{aligned}
(2\sigma-1)\sum_{\gamma}\frac{x^{i\gamma}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}&=-x^{-\frac{1}{2}}\left(\sum_{n\le x}\Lambda(n)\left(\frac{x}{n}\right)^{1-\sigma+it}+\sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{\sigma+it}\right)\\
&\quad-x^{\frac{1}{2}-\sigma+it}\frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)}+\frac{x^{\frac{1}{2}}(2\sigma-1)}{(\sigma-1+it)(\sigma-it)}\\
&\quad-x^{-\frac{1}{2}}\sum_{n=1}^{\infty}\left(\frac{(2\sigma-1)x^{-2n}}{(\sigma-1-2n-it)(\sigma+2n+it)}\right).
\end{aligned}
\tag{2.8}
$$

By continuity (2.8) holds for all $x>1$. Therefore we no longer exclude the values $x=1$ and $x=p^n$. We now bound the last three terms on the right hand-side of equation (2.8). Riemann proved the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

from which

$$\zeta(s)=\frac{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}. \tag{2.9}$$

We use Legendre's duplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$$

and change s into $\frac{1-s}{2}$ to obtain

$$\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s}{2}+\frac{1}{2}\right)=2^{1-2\left(\frac{1-s}{2}\right)}\pi^{\frac{1}{2}}\Gamma\left(2\left(\frac{1-s}{2}\right)\right),$$

so that

$$\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)=2^{s}\pi^{\frac{1}{2}}\Gamma\left(1-s\right). \tag{2.10}$$

We use the functional relation

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$$

and change s into $\frac{s}{2}$ to obtain

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)=\frac{\pi}{\sin\frac{\pi s}{2}}. \tag{2.11}$$

Dividing (2.10) by (2.11) and simplifying, we get

$$\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}=2^{s}\pi^{-\frac{1}{2}}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(1-s\right).$$

Inserting this into (2.9), we obtain

$$\zeta(s)=2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(1-s\right)\zeta(1-s). \tag{2.12}$$

Changing s into $s-1$, the functional equation becomes

$$\zeta(1-s)=2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma\left(s\right)\zeta(s).$$

Taking logarithms,

$$\log\zeta(1-s)=\log 2-s\log 2\pi+\log\left(\cos\frac{\pi s}{2}\right)+\log\left(\Gamma\left(s\right)\right)+\log\zeta(s) \tag{2.13}$$

and differentiating this,

$$-\frac{\zeta'(1-s)}{\zeta(1-s)}=-\log 2\pi-\frac{\pi}{2}\tan\frac{\pi s}{2}+\frac{\Gamma'(s)}{\Gamma(s)}+\frac{\zeta'(s)}{\zeta(s)}.$$

Putting $s = \sigma - it$, we obtain

$$-\frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)} = \log 2\pi - \frac{\pi}{2}\tan\frac{\pi(\sigma-it)}{2} + \frac{\Gamma'(\sigma-it)}{\Gamma(\sigma-it)} + \frac{\zeta'(\sigma-it)}{\zeta(\sigma-it)}.$$

However, it is well-known that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|)^{-1}$$

and

$$\frac{\pi}{2}\tan\frac{\pi s}{2} = -\frac{1}{s-1} + O(|s-1|).$$

Thus, we have

$$\log 2\pi + \frac{\pi}{2}\tan\frac{\pi(\sigma-it)}{2} - \frac{\Gamma'(\sigma-it)}{\Gamma(\sigma-it)} = \log\tau + O_\sigma(1),$$

and hence

$$\frac{\zeta'(1-\sigma+it)}{\zeta(1-\sigma+it)} = -\frac{\zeta'(\sigma-it)}{\zeta(\sigma-it)} - \log\tau + O_\sigma(1) = -\log\tau + O_\sigma(1),$$

since

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\Lambda(n) < \frac{1}{\sigma-1} + A,$$

for $1 < \sigma \le 2$, where A denotes a positive absolute constant.

We observe in (2.8) that

$$\frac{x^{\frac{1}{2}}(2\sigma-1)}{(\sigma-1+it)(\sigma-it)} = O_\sigma\left(x^{\frac{1}{2}}\tau^{-1}\right)$$

and

$$x^{-\frac{1}{2}}\sum_{n=1}^{\infty}\frac{(2\sigma-1)x^{-2n}}{(\sigma-1-2n-it)(\sigma+2n+it)} = O_\sigma\left(x^{-\frac{1}{2}-2}\tau^{-1}\right) = O_\sigma\left(x^{-\frac{5}{2}}\tau^{-1}\right).$$

Collecting all of the above equations in (2.8), we find that

$$\begin{aligned}(2\sigma-1)\sum_\gamma \frac{x^{i\gamma}}{(\sigma-\frac{1}{2})^2(t-\gamma)^2} &= -x^{-\frac{1}{2}}\left(\sum_{n\le x}\Lambda(n)\left(\frac{x}{n}\right)^{1-\sigma+it} + \sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{\sigma+it}\right) \\ &\quad - x^{\frac{1}{2}-\sigma+it}(-\log\tau + O_\sigma(1)) + O_\sigma\left(x^{\frac{1}{2}}\tau^{-1}\right) + O_\sigma(x^{-\frac{5}{2}}\tau^{-1}) \\ &= -x^{-\frac{1}{2}}\left(\sum_{n\le x}\Lambda(n)\left(\frac{x}{n}\right)^{1-\sigma+it} + \sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{\sigma+it}\right) \\ &\quad + x^{\frac{1}{2}-\sigma+it}(\log\tau + O_\sigma(1)) + O_\sigma\left(x^{\frac{1}{2}}\tau^{-1}\right),\end{aligned}$$

for $1 < \sigma \le 2$, $x \ge 2$, which is the desired result. This concludes the proof.

2.2 Pair Correlation of Zeros of $\zeta(s)$

In order to study the distribution of the differences $\gamma - \gamma'$ between successive zeros of $\zeta(s)$, Montgomery defined, for real α and $T \geq 2$, the pair-correlation function

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0\leq\gamma,\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'), \tag{2.14}$$

where $w(u) = \frac{4}{4+u^2}$ is a suitable weighting function. Montgomery's result for the pair-correlation function $F(\alpha)$ can be summerized as follows

Theorem 1. *(Assume RH.) For real α and $T \geq 2$,*

(i) *$F(\alpha)$ is real and $F(\alpha) = F(-\alpha)$.*

(ii) *If $T > T_0(\epsilon)$, then $F(\alpha) \geq -\epsilon$ for all α.*

(iii) *For fixed α such that $0 \leq \alpha \leq 1$,*

$$F(\alpha) = (1 + o(1))T^{-2\pi}\log(T) + \alpha + o(1),$$

as T tends to infinity, uniformly for $0 \leq \alpha \leq 1 - \epsilon$.

We shall prove the three assertions of Theorem 1 below. The key to the proof lies in the explicit formula in Lemma 1. The basic idea is to express the pair correlation function $F(\alpha)$ in terms of the square integral of one of the terms of this explicit formula, so that $F(\alpha)$ is bounded from below. Then it remains to bound the square of the integral of the remaining terms of the explicit formula.

We proceed to prove Theorem 1. To prove (i), for real α and $T \geq 2$, we interchange the roles of γ and γ' and observe that $w(u)$ is an even function of u, to conclude that $F(\alpha) = F(-\alpha)$. Then

$$\overline{F(\alpha)} = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leq T} T^{-i\alpha(\gamma-\gamma')} w(\gamma-\gamma') = F(-\alpha) = F(\alpha)$$

Hence, $F(\alpha)$ must be real.

To prove (ii), we take $\sigma = \frac{2}{3}$ in (2.1), so that for $x \geq 1$

$$2\sum_{\gamma} \frac{x^{i\gamma}}{1+(t-\gamma)^2} = -x^{-\frac{1}{2}} \left(\sum_{n\leq x} \Lambda(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{n>x} \Lambda(n) \left(\frac{x}{n}\right)^{\frac{3}{2}+it} \right) + x^{-1+it} \left(\log\tau + O_\sigma(1)\right) + O_\sigma\left(x^{\frac{1}{2}}\tau^{-1}\right). \tag{2.15}$$

Montgomery detects a cancelation from $x^{i\gamma}$ by noting that the weight of the sum over γ concentrates the sum to the γ's in a short bounded interval around t, so that the sum's behavior is roughly that of

$$\sum_{t<\gamma<t+1} x^{i\gamma} \sim \log t$$

which is much smaller than $\log t$, because

$$N(T+1) - N(T) = \sum_{T<\gamma\leq T+1} 1 \ll \log T.$$

If x is close to 1, then more cancelation can occur. Since the sum over primes is concentrated around x, it behaves like

$$\sum_{\frac{x}{2}<n<2x} \frac{\Lambda(n)}{n^{it}}.$$

We denote (2.15) by

$$L(x,t) = R(x,t) \tag{2.16}$$

and square the absolute value of the sum over zeros. We have

$$\begin{aligned}\int_0^T |L(x,t)|^2\,dt &= \int_0^T \left| \sum_{0<\gamma\leq T} \frac{x^{i\gamma}}{1+(t-\gamma)^2} \right|^2 dt \\ &= 4\int_0^T \sum_{0<\gamma,\gamma'\leq T} x^{i(\gamma-\gamma')} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma\prime)^2)} \\ &\quad + 8\int_0^T \sum_{\substack{\gamma,\gamma' \\ \gamma\notin[0,1)}} x^{i(\gamma-\gamma')} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)}.\end{aligned}$$

To examine this, we make use of the following results from Davenport's classical book [3] (pp. 98–99, Formula (3)): *For $t \geq T$,*

$$\sum_{\gamma>T} \frac{1}{1+(t-\gamma)^2} \ll \left(\frac{1}{t+1} + \frac{1}{T-t+1}\right) \log T$$

and

$$\sum_{\gamma} \frac{1}{1+(t-\gamma)^2} \ll \log T.$$

Since $\left|x^{i(\gamma-\gamma')}\right| = 1$ for all real x,

$$\int_0^T \sum_{\substack{\gamma,\gamma' \\ \gamma\notin[0,T)}} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} \ll \int_0^T \left(\frac{1}{t+1} + \frac{1}{T-t+1}\right) \log T \log T \, dt \ll \log^3 T,$$

and hence

$$\int_0^T |L(x,t)|^2 \, dt = 4 \int_0^T \sum_{0<\gamma,\gamma'\leq T} x^{i(\gamma-\gamma')} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} + O(\log^3 T)$$

We can extend the range of integration on the right side from $(0,T]$ to $(\infty,-\infty)$ with a small error. To accomplish this, we need a third result, which is an immediate consequence of the above two results: *For* $t \geq T$,

$$\sum_{0<\gamma\leq T} \frac{1}{1+(t-\gamma)^2} \ll \frac{1}{t-T+1} \log T.$$

From this, we have

$$\int_T^\infty \sum_{0<\gamma,\gamma'\leq T} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} \ll \log^2 T \int_T^\infty \frac{dt}{|t-Y+1|^2} \ll \log^2 T$$

and

$$\int_{-\infty}^0 \sum_{0<\gamma,\gamma'\leq T} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} \ll \log^2 T \int_{-\infty}^0 \frac{dt}{|t-Y+1|^2} = o(\log^2 T).$$

Hence,

$$\begin{aligned} \int_0^T |L(x,t)|^2 \, dt &= 4 \left(\int_{-\infty}^0 + \int_0^T + \int_T^\infty\right) \sum_{0<\gamma,\gamma'\leq T} x^{i(\gamma-\gamma')} \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} \\ &\quad + O(\log^3 T) \\ &= 4 \sum_{0<\gamma,\gamma'\leq T} x^{i(\gamma-\gamma')} \int_{-\infty}^\infty \frac{dt}{(1+(t-\gamma)^2)(1+(t-\gamma')^2)} + O(\log^3 T), \end{aligned} \tag{2.17}$$

and we evaluate the integral on the far right-hand side of (2.17) using the calculus of residues.

Let

$$f(z) = \frac{1}{(1+(z-\gamma)^2)(1+(z-\gamma')^2)}.$$

Let $\rho > 0$ and let Γ be the simple closed contour defined by $\Gamma = \gamma_\rho \cup C_\rho^+$, where γ_ρ is the line segment given by $\gamma_\rho = [-\rho, \rho]$ and where C_ρ^+ is the upper half semi-circle parametrized by $z = \rho e^{i\theta}$, where $0 \leq \theta \leq \pi$. We have f holomorphic in Γ with two simple poles at $z = \gamma + i$ and $z = \gamma' + i$. We have

$$\int_\Gamma f(z)\,dz = \int_{\gamma_\rho} f(z)\,dz + \int_{C_\rho} f(z)\,dz = \int_{-\rho}^{\rho} f(z)\,dz + \int_{C_\rho} f(z)\,dz.$$

Letting ρ go to infinity,

$$\lim_{\rho\to\infty} \int_\Gamma f(z)\,dz = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} f(z)\,dz + \lim_{\rho\to\infty} \int_{C_\rho^+} f(z)\,dz.$$

On noting that

$$\begin{aligned} \left| \int_{C_\rho^+} f(z) dz \right| &\leq \int_0^\pi \frac{\left| i\rho e^{i\theta} \right| |\,d\theta|}{|1+(\rho e^{i\theta}-\gamma)^2|\,|1+(\rho e^{i\theta}-\gamma\prime)^2|} \\ &\leq \frac{\pi\rho}{((\rho-\gamma)^2-1)((\rho-\gamma')^2-1)}, \end{aligned}$$

which tends to infinity, we have

$$\int_\Gamma f(z)\,dz = \int_{-\infty}^{\infty} f(z)\,dz. \tag{2.18}$$

By Cauchy's residue theorem,

$$\begin{aligned} \int_\Gamma f(z)\,dz = 2\pi i \Bigg(&\lim_{z\to\gamma+i} \frac{z-(\gamma+i)}{(1+(z-\gamma)^2)\,(1+(z-\gamma')^2)} \\ &+ \lim_{z\to\gamma'+i} \frac{z-(\gamma'+i)}{(1+(z-\gamma)^2)\,(1+(z-\gamma')^2)} \Bigg). \end{aligned} \tag{2.19}$$

The residue of f at $z = \gamma + i$ is

$$\begin{aligned} \lim_{z\to\gamma+i} (z-(\gamma+i)) f(z) &= \lim_{z\to\gamma+i} \frac{z-(\gamma+i)}{(1+(z-\gamma)^2)\,(1+(z-\gamma')^2)} \\ &= \lim_{z\to\gamma+i} \frac{-1}{(1-i(z-\gamma))(z-(\gamma'+i))(1-(z-\gamma'))} \\ &= \frac{-1}{2(\gamma-\gamma')(2-i(\gamma-\gamma'))} \end{aligned}$$

and the residue of $f(z)$ at $z = \gamma' + i$ is, by symmetry,

$$\lim_{z\to\gamma'+i}(z - (\gamma' + i))f(z) = \frac{-1}{2(\gamma' - \gamma)(2 - i(\gamma' - \gamma))}.$$

Substituting these results into (2.19),

$$\begin{aligned}\int_\Gamma f(z)dz &= \pi i\left(\frac{2 + i(\gamma - \gamma') - 2 + i(\gamma - \gamma')}{-(\gamma - \gamma')(2 - i(\gamma - \gamma'))(2 + i(\gamma - \gamma'))}\right)\\ &= \frac{2\pi}{(2 - i(\gamma - \gamma'))(2 + i(\gamma - \gamma'))}\\ &= \frac{2\pi}{4 + (\gamma - \gamma')^2}\end{aligned} \tag{2.20}$$

By (2.18) and (2.20),

$$\int_{-\infty}^{\infty}\frac{dt}{(1 + (t - \gamma)^2)(1 + (t - \gamma')^2)} = \frac{\pi}{2}w(\gamma - \gamma'),$$

and inserting this into the far right-hand side of (2.17) we obtain

$$\int_0^T |L(x,t)|^2\, dt = 2\sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')}w(\gamma - \gamma') + O(\log^3 T).$$

The explicit formula (2.15) holds for $x \ge 1$. However, the function $L(x,t)$ is defined for all real numbers $x \ge 0$ only. We follow Montgomery and set $x = T^\alpha$ for α and $T \ge 2$ to obtain

$$\begin{aligned}\int_0^T |L(T^\alpha,t)|^2\, dt &= 2\pi\sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')}w(\gamma - \gamma') + O(\log^3 T)\\ &= F(\alpha)T\log T + O(\log^3 T),\end{aligned}$$

using (2.13). Since the integral is nonnegative, $F(\alpha)T\log T + O(\log^3 T) \ge 0$ for all α, so that $F(\alpha)T\log T \ge -O(\log^3 T)$, from which $F(\alpha) \ge -O\left(\frac{\log^3 T}{T\log T}\right)$. We set $T = T(\epsilon)$ such that $F(\alpha) = F(\alpha, T) \ge -\epsilon$.

To prove (iii), we must examine $\int_0^T |R(x,t)|^2\, dt$, which is equal to

$$\int_0^T\left|-x^{-\frac{1}{2}}\left(\sum_{n\le x}\Lambda(n)\left(\frac{x}{n}\right)^{-\frac{1}{2}+it}+\sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{\frac{3}{2}-it}\right)+x^{-1+it}\log\tau+x^{-1+it}O_\sigma(1)+O_\sigma(x^{\frac{1}{2}}\tau^{-1})\right|^2 dt$$

and bound the square integral of each of the four terms in $R(x,t)$. For $x \ge 1$ and $T \ge 2$,

$$\begin{aligned}\int_0^T \left|x^{-1+it}\log\tau\right|^2\, dt &= \int_0^T \left|x^{-1+it}\right|^2 |\log(|t| + 2)|^2\, dt\\ &= x^{-2}\int_0^{T+2}(\log u)^2\, du,\end{aligned}$$

where $u = t + 2$. By partial integration,

$$\int (\log u)^2 du = u(\log u)^2 - 2(u \log u - u)\, du,$$

from which

$$\int_0^T |x^{-1+it} \log \tau|^2 \, dt = x^{-2} T \log T (\log T + O(1)).$$

We have

$$\int_0^T \left| x^{-1+it} O_\sigma(1) \right|^2 \, dt = O_\sigma(x^{-2} T)$$

and

$$\int_0^T \left| O_\sigma(x^{\frac{1}{2}} \tau^{-1}) \right|^2 \, dt = O_\sigma(x) \int_0^T \left| \tau^{-1} \right|^2 \, dt = O_\sigma\left(\frac{x}{T}\right) = O_\sigma(x),$$

if $T \geq 2$. Finally, to bound

$$\begin{aligned} &\int_0^T \left| -x^{-\frac{1}{2}} \left(\sum_{n \leq x} \Lambda(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{n > x} \Lambda(n) \left(\frac{x}{n}\right)^{\frac{3}{2}-it} \right) \right|^2 dt \\ &\qquad = -x^{-1} \int_0^T \left| \left(\sum_{n \leq x} \Lambda(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{n > x} \Lambda(n) \left(\frac{x}{n}\right)^{\frac{3}{2}-it} \right) \right|^2 dt \end{aligned} \tag{2.21}$$

we use a version of Hilbert's inequality due to Montgomery and Vaughn:

$$\int_0^T \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{it}} \right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 \left(T + O(n)\right).$$

Applying this to the right hand-side of (2.21), we obtain

$$\begin{aligned} &\int_0^T \left| -x^{-\frac{1}{2}} \left(\sum_{n \leq x} \Lambda(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{n > x} \Lambda(n) \left(\frac{x}{n}\right)^{\frac{3}{2}-it} \right) \right|^2 dt \\ &= x^{-1} \sum_{n \leq x} \Lambda^2(n) \left(\frac{x}{n}\right)^{-1} (T + O(n)) + x^{-1} \sum_{n > x} \Lambda^2(n) \left(\frac{x}{n}\right)^3 (T + O(n)) \\ &= x^{-2} T \sum_{n \leq x} \Lambda^2(n) n + x^{-2} \sum_{n \leq x} \Lambda^2(n) n O(n) + x^2 T \sum_{n > x} \Lambda^2(n) n^{-3} + x^2 \sum_{n > x} \Lambda^2(n) n^{-3} O(n). \end{aligned}$$

Our goal is to show that

$$\frac{1}{x} \int_0^T \left| \sum_{n \leq x} \Lambda(n) \left(\frac{x}{n}\right)^{-\frac{1}{2}+it} + \sum_{n > x} \Lambda(n) \left(\frac{x}{n}\right)^{\frac{3}{2}+it} \right|^2 dt = T \left(\log x + O(1)\right) + O(x \log x).$$

We shall use a general form of Parseval's identity for Dirichlet series:

$$\int_0^T \left|\sum_n a_n n^{-it}\right|^2 = \sum_n |a_n|^2 \left(T + O(n)\right).$$

From this, we obtain

$$\begin{aligned} M_1 &= \frac{1}{x}\sum_{n\le x}\Lambda^2(n)\left(\frac{x}{n}\right)^{-1}(T+O(n)) + \frac{1}{x}\sum_{n>x}\Lambda^2(n)\left(\frac{x}{n}\right)^3(T+O(n)) \\ &= Tx^{-2}\sum_{n\le x}\Lambda^2(n)n + x^{-2}\sum_{n\le x}\Lambda^2(n)nO(n) + Tx^2\sum_{n>x}\Lambda^2(n)\frac{1}{n^3} + x^2\sum_{n>x}\Lambda^2(n)\frac{O(n)}{n^3}. \end{aligned}$$

Let $N_x = \left[\frac{\log x}{\log 2}\right]$. Since

$$\sum_{p\le x} p\log^2 p = \frac{1}{2}x^2\log x + O(x^{\frac{3}{2}}\log^3 x),$$

we have

$$\begin{aligned} Tx^{-2}\sum_{n\le x}\Lambda^2(n)n &= Tx^{-2}\sum_{k=1}^{N_x}\sum_{p^k\le x} p\log^2 p \\ &= Tx^{-2}\sum_{p\le x} p\log^2 p + Tx^{-2}\sum_{k=1}^{N_x}\sum_{p\le x^{\frac{1}{k}}} p\log^2 p \\ &= Tx^{-2}\left(\frac{1}{2}x^2\log x + O(x^3 2 + \log^3 x)\right) + Tx^{-2}\sum_{k=1}^{N_x}\left(\frac{1}{2k}x^{\frac{2}{k}}\log x + O(x^3 2 + \log^3 x)\right) \\ &\ll \frac{1}{2}T\log x + O(Tx^{-\frac{1}{2}}\log^3 x) + O(Tx^{-1}\log^2 x) + O(Tx^{-\frac{5}{4}}\log^4 x) \\ &= \frac{1}{2}T\log x + O(T). \end{aligned}$$

Next, since

$$\sum_{p\le x} p^2\log^2 p = x^3\log x(1+O(1)),$$

we have

$$\begin{aligned} x^{-2}\sum_{n\le x}\Lambda^2(n)nO(n) &= x^{-2}\sum_{k=1}^{N_x}\sum_{p^k\le x} p^2\log^2 p \\ &= x^{-2}\sum_{p\le x} p^2\log^2 p + x^{-2}\sum_{k=2}^{N_x}\sum_{p\le x^{\frac{1}{k}}} p^2\log^2 x \\ &\ll x\log x(1+O(1)) + x^{-\frac{1}{2}}\log^2 x(1+O(1)) \\ &\ll x\log x. \end{aligned}$$

Since the primes $\gg x$ do not contribute, we consider only truncated versions of

$$Tx^2 \sum_{n>x} \Lambda^2(n) \frac{1}{n^3}$$

and

$$x^2 \sum_{n>x} \Lambda^2(n) \frac{O(n)}{n^3}.$$

We shall make use of the following well-known results:

$$\sum_{p \le x} \frac{\log p}{p} = \log x + 1 + O(x^{-\frac{1}{2}} \log^2 x)$$

and

$$\sum_{p \le x} \frac{\log^2 p}{p} = \frac{1}{2} \frac{\log x}{x^2} + O(x^{-2}).$$

The first powers of primes contribute

$$Tx^2 \sum_{p=x}^{x^2} \frac{\log^2 p}{p^3} = Tx^2 \left(\frac{1}{2} \frac{\log x}{x^2} + O(x^{-2}) \right) = \frac{1}{2} T \log x + O(T),$$

and the higher powers of primes contribute

$$Tx^2 \sum_{k=2}^{N_x} \sum_{p=x}^{x^2} \frac{\log^2 p}{p^{3k}} \ll Tx^2 \log^2 x \sum_{k=2}^{N_x} \sum_{p^k=x}^{x^2} p^{\frac{1}{2}k} \le Tx^2 \log^2 x \sum_{k=2}^{N_x} \sum_{p^k=x}^{x^2} \frac{1}{p^6}.$$

To estimate the inner sum, we write

$$Tx^2 \sum_{k=2}^{N_x} \sum_{p=x}^{x^2} \frac{\log^2 p}{p^{3k}} \le Tx^2 \log^2 x \sum_{k=2}^{N_x} \sum_{p^k=x}^{x^2} \frac{1}{p^6} \sum_{n=x^{\frac{1}{2}}}^{x} \frac{1}{n^6} \ll Tx^2 \log^3 x \left(x^{-\frac{5}{2}} \right) \ll T.$$

Hence

$$Tx^2 \sum_{n>x} \Lambda^2(n) \frac{1}{n^3} = \frac{1}{2} T \log x + O(T).$$

To bound

$$x^2 \sum_{n>x} \Lambda^2(n) \frac{O(n)}{n^3},$$

we shall make use of

$$\sum_{p=x}^{x^2} \frac{\log^2 p}{p^2} = O(x^{-1} \log x).$$

The contribution of the first powers of primes is

$$x^2 \sum_{p=x}^{x^2} \log^2 p O\left(\frac{1}{p^2}\right) = x^2 O(x^{-1}\log x) = O(x \log x),$$

and the higher powers of primes contribute

$$x^2 \sum_{k=2}^{N_x} \sum_{p^k}^{x^2} \log^2 p O\left(\frac{1}{p^{2k}}\right) \ll x^2 \log^2 x \sum_{k=2}^{N_x} \sum_{p^k}^{x^2} \frac{1}{p^{2k}}.$$

Here, we note that when $k = 2$

$$x^2 \sum_{k=2}^{N_x} \sum_{p^k}^{x^2} \frac{1}{p^{2k}} \ll x^2 \log^2 x \left[\frac{2\log x}{\log 2}\right] \sum_{n=x^{\frac{1}{2}}}^{x} \frac{1}{u^4} \ll x^{\frac{1}{2}} \log^3 x.$$

Hence,

$$x^2 \sum_{n>x} \Lambda^2(n) \frac{O(n)}{n^3} = O(x \log x),$$

and assembling all bounds, we obtain

$$M_1 = T(\log x + O(1)) + O(x \log x).$$

To finish the proof of (iii), we shall make use of the Cauchy-Scharz inequality in the form: *We have*

$$\int_0^T |f_k(t)|^2 \, dt = A_k.$$

for $1 \leq k \leq n$. *Suppose* $A_n \leq A_{n-1} \leq \cdots \leq A_2 \leq A_1$, *then*

$$\int_0^T \left|\sum_{k=1}^{n} f_k(t)\right|^2 \, dt = A_1 + O((A_1 A_2)^{\frac{1}{2}}).$$

We obtain the relative magnitude of M_1, M_2, M_3, M_4, where

$$\begin{aligned}
M_1 &= T(\log x + O(1)), \\
M_2 &= \frac{T}{x^2} \log^2 T + O(\log T), \\
M_3 &= O\left(\frac{T}{x^2}\right), \\
M_4 &= O(x).
\end{aligned}$$

We distinguish three cases. First, we suppose that $1 \le x \le \log^{\frac{3}{4}} T$ and note that

$$M_2 = \frac{T}{x^2}\left(\log^2 T + O(\log T)\right) \ge \frac{T}{\log^{\frac{1}{2}} T}\log^2 T + T\frac{O(\log T)}{\log^{\frac{3}{2}} T} = O(T\log^{\frac{1}{2}} T),$$

while

$$\begin{aligned}
M_1 &\le T\log\left(\log^{\frac{3}{4}} T\right) + O(T) + O(\log^{\frac{3}{4}} T\log(\log^{\frac{3}{4}} T)) = o(M_2),\\
M_3 &= O\left(\frac{T}{\log^{\frac{3}{2}} T}\right) = o(M_2),\\
M_4 &= O\left(\frac{T}{\log^{\frac{3}{2}} T}\right) = o(M_2).
\end{aligned}$$

Second, when $\log^{\frac{3}{4}} T < x \le \log^{\frac{3}{2}} T$, all four terms are $o(T\log T)$. Third, when $\log^{\frac{3}{2}} T < x < \frac{T}{\log T}$, we see that M_1 dominates with all the terms being $o(M_1)$. We apply the Cauchy-Schwarz inequality to each of the three ranges individually. In the first range, we have

$$\int_0 T\left|R(x,t)\right|^2 dt = M_2 + O\left((M_1M_2)^{\frac{1}{2}}\right) = (1+o(1))\frac{T}{x^2}\log^2 T.$$

In the second range, we have $M_1 = M_2 = M_3 = M_4 = o(T\log T)$, so that

$$\int_0^T \left|R(x,t)\right|^2 dt = M_2 + O\left((M_1M_2)^{\frac{1}{2}}\right) = o(T\log T) = 0.$$

In the third range, we have

$$M_1 = T(\log x + O(1)) + O(x\log x),$$

and so

$$\int_0^T \left|R(x,t)\right|^2\, dt = M_1 + O((M_1M_2)^{\frac{1}{2}}) = (1+o(1))T\log x.$$

All together, we have

$$\int_0^T \left|R(x,t)\right|^2\, dt = (1+o(1))\frac{T}{x^2}\log^2 T + o(T\log T) + (1+o(1))T\log x,$$

for $1 \le x < \frac{T}{\log T}$.

Let now $x = T^\alpha$ for any $0 \le \alpha \le 1-\epsilon$. We have

$$\begin{aligned}
\int_0^T \left|R(x,t)\right|^2 dt &= (1+o(1))\frac{T}{T^{2\alpha}}\log^2 T + o(T\log T) + (1+o(1))T\log T^\alpha\\
&= ((1+o(1))T^{-2\alpha}\log T + \alpha + o(1))T\log T,
\end{aligned}$$

uniformly for $0 \leq \alpha \leq 1-\epsilon$. Using $L(x,t)=R(x,t)$, we get

$$\int_0 T \left|L(x,t)\right|^2 \, dt = \int_0^T \left|R(x,t)\right|^2 \, dt,$$

where $x = T^\alpha$. Letting T tend to infinity, we obtain

$$F(\alpha)T\log T + O(\log^3 T) = ((1+o(1))T^{-2\alpha}\log T + \alpha + o(1))T\log T,$$

from which follows

$$F(\alpha) = (1+o(1))T^{-2\alpha}\log T + \alpha + o(1).$$

2.3 Applications to Simple Zeros and Prime Numbers

Montgomery's theorem leads to several corollaries from the definition of $F(\alpha,T)$. Their proofs make use of a certain convolution formula. By definition,

$$F(\alpha,T) = \left(\frac{T}{2\pi}\right)^{-1} \sum_{\alpha,\alpha'\in[0,T]} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma').$$

Assuming Fourier inversion holds, we have

$$\begin{aligned}
\frac{T}{2\pi}\log T \int_{-\infty}^{\infty} F(\alpha,T)\hat{r}(\alpha)d\alpha &= \int_{-\infty}^{\infty} \sum_{\alpha,\alpha'\in[0,T]} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma')\hat{r}\alpha d\alpha \\
&= \sum_{\alpha,\alpha'\in[0,T]} w(\gamma-\gamma') \int_{-\infty}^{\infty} T^{i\alpha(\gamma-\gamma')}\hat{r}(\alpha)d\alpha \\
&= \sum_{\alpha,\alpha'\in[0,T]} w(\gamma-\gamma') \int_{-\infty}^{\infty} \hat{r}(\alpha)e^{\frac{2\pi i\alpha(\gamma-\gamma')\log T}{2\pi}} d\alpha \\
&= \sum_{\alpha,\alpha'\in[0,T]} r\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma').
\end{aligned}$$

Corollary 1. *(Assume RH.) For fixed $0<a<1$,*

$$\sum_{\alpha,\alpha'\in[0,T]} \left(\frac{\sin(a(\gamma-\gamma')\log T)}{a(\gamma-\gamma')\log T}\right) w(\gamma-\gamma') \sim \left(\frac{1}{2a}+\frac{a}{2}\right)\frac{T}{2\pi}\log T$$

and

$$\sum_{\alpha,\alpha'\in[0,T]} \left(\frac{\sin((a/2)(\gamma-\gamma')\log T)}{(a/2)(\gamma-\gamma')\log T}\right) w(\gamma-\gamma') \sim \left(\frac{1}{a}+\frac{a}{3}\right)\frac{T}{2\pi}\log T$$

as $T\to\infty$.

Proof. Fixed $0 < a < 1$. By Fourier inversion,

$$r_1(u) = \int_{-\infty}^{\infty} \hat{r}_1(\xi) e^{2\pi i u \xi} d\xi = \frac{1}{2a} \int_{-a}^{a} e^{2\pi i u \xi} d\xi = \frac{\sin(2\pi u a)}{2\pi u a}.$$

Hence, $r_1(u)$ has the Fourier Transform $\hat{r}_1(\xi)$, where χ_a denotes the characteristic function of the interval $[-a, a]$. Furthermore, the function

$$s(u) = \left(\frac{\sin \pi u}{\pi u} \right)^2$$

has the Fourier transform

$$\hat{s}(\xi) = (1 - |\xi|) \chi_1(\xi),$$

since

$$s(u) = \int_{-\infty}^{\infty} \hat{s}(\xi) e^{2\pi i u \xi} d\xi = \int_{-1}^{1} (1 - |\xi|) e^{2\pi i u \xi} d\xi = \frac{2}{(2\pi i u)^2} (\cos 2\pi u - 1) = \left(\frac{\sin \pi u}{\pi u} \right)^2.$$

Likewise, the function

$$r_2(u) = \left(\frac{\sin \pi a v u}{\pi a u} \right)^2$$

has the Fourier Transform

$$\hat{r}_2(\xi) = \frac{1}{a^2} (a - |\xi|) \chi_a(\xi),$$

since

$$\hat{s}\left(\frac{\xi}{a} \right) = \left(1 - \frac{|\xi|}{a} \right) \chi_a(\xi) = a \hat{r}_2(\xi)$$

and

$$r_2(u) = \int_{-\infty}^{\infty} \hat{r}_2(\xi) e^{2\pi i u \xi} d\xi = \int_{-\infty}^{\infty} \hat{s}(\xi) e^{2\pi i u a \xi} d\xi = s(au) = \left(\frac{\sin \pi a u}{\pi a u} \right)^2.$$

We are now ready to prove Corollary 1. We have

$$\sum_{\alpha, \alpha' \in [0,T]} \left(\frac{\sin (2\pi a (\gamma - \gamma')(\log T)/2\pi)}{2\pi a (\gamma - \gamma')(\log T)/2\pi} \right) w(\gamma - \gamma') = \frac{T}{2\pi} \log T + \int_{-\infty}^{\infty} F(u, T) \hat{r}_1(u) \, du \quad (2.22)$$

and compute that, as $T \to \infty$,

$$\begin{aligned} \int_{-\infty}^{\infty} F(u, T) \hat{r}_2(u) \, du &= \frac{1}{2a} \int_{-\infty}^{\infty} F(u, T) \, du \\ &= \frac{1}{a} \int_{0}^{a} \left((1 + o(1)) T^{-2u} \log T + u + o(1) \right) \, du \\ &= \frac{1}{2a} + \frac{a}{2}. \end{aligned}$$

Together with (2.22), this gives

$$\sum_{\alpha,\alpha'\in[0,T]} \left(\frac{\sin(a(\gamma-\gamma')\log T)}{a(\gamma-\gamma')\log T}\right) w(\gamma-\gamma') \sim \left(\frac{1}{2a}+\frac{a}{2}\right)\frac{T}{2\pi}\log T.$$

To prove the second formula in the corollary, we note that

$$\sum_{\alpha,\alpha'\in[0,T]} \left(\frac{\sin((a/2)(\gamma-\gamma')\log T)}{(a/2)(\gamma-\gamma')\log T}\right) w(\gamma-\gamma') = \frac{T}{2\pi}\log T \int_{-\infty}^{\infty} F(u,T)\hat{r}_2(u)\,du,$$

where

$$\begin{aligned}\int_{-\infty}^{\infty} F(u,T)\hat{r}_2(u)du &= \frac{1}{a^2}\int_{-\infty}^{a} F(u,T)(a-|u|)du \\ &= \frac{2}{a}\int_0^a F(u,T)du - \frac{2}{a^2}\int_0^a uF(u,T)du\end{aligned} \tag{2.23}$$

However, as $T\to\infty$,

$$\frac{2}{a}\int_0^a F(u,T)\,du = \frac{2}{a}\int_0^a u\left[(1+o(1))T^{-2u}\log T + u + o(1)\right]\,du = \frac{1}{a}+a \tag{2.24}$$

and

$$\frac{2}{a^2}\int_0^a uF(u,T)\,du = \frac{2}{a^2}\int_0^a u\left[(1+o(1))T^{-2u}\log T + u + o(1)\right]\,du = \frac{2a}{3} \tag{2.25}$$

Inserting (2.24) and (2.25) into (2.23), we obtain

$$\int_{-\infty}^{\infty} F(u,T)\hat{r}_2(u)\,du = \frac{1}{a}+\frac{a}{3}. \tag{2.26}$$

and hence we have the corollary. □

The second statement of Montegomery's Corollary 1 leads to an interesting result which is the focus of my thesis.

Corollary 2. *(Assume RH.) As* $T\to\infty$,

$$\sum_{\substack{\gamma\in[0,T]\\ \rho \text{ simple}}} 1 \geq \left(\frac{2}{3}+o(1)\right)\frac{T}{2\pi}\log T,$$

where the sum is taken over the height of all simple zeros $\rho=\frac{1}{2}+i\gamma$ *of* $\zeta(s)$ *with height* γ *within the interval* $[0,T]$ *on the critical line.*

Proof. For each zero ρ let m_ρ denote multiplicity of the zero. We have

$$\sum_{\substack{\gamma,\gamma'\in[0,T]\\ \gamma=\gamma'}} 1 = \sum_{\gamma\in[0,T]} m_\rho.$$

Each zero ρ of multiplicity of m_ρ is counted m_ρ^2 times on each side. On the left-hand side, for a fixed zero ρ, there are m_ρ heights γ of equal value. There are m_ρ^2 ways to choose an ordered pair (γ,γ') with $\gamma=\gamma'$, so that ρ is counted m_ρ times. On the right-hand side, for fixed zero ρ, m_ρ is added once for each of the m_ρ identical heights associated with the zero ρ. Since $\sin x \le a$ for $0 \le x \le \frac{\pi}{4}$ we have $\frac{\sin s}{s} \approx 1$ for x sufficiently small. Using this and the fact that $w(0)=1$, we have for some fuxed $0<a<1$ and each $T\ge 2$

$$\sum_{\substack{\gamma,\gamma'\in[0,T]\\ \gamma=\gamma'}} 1 \approx \sum_{\substack{\gamma,\gamma'\in[0,T]\\ \gamma=\gamma'}} \left(\frac{\sin((a/2)(\gamma-\gamma')\log T)}{(a/2)(\gamma-\gamma')\log T}\right)^2 w(\gamma-\gamma')$$

adding the right-hand side the terms of the nonidentical pairs γ and γ' will increase the value of the sum. Thus, we have

$$\sum_{\substack{\gamma,\gamma'\in[0,T]\\ \gamma=\gamma'}} 1 \le \sum_{\alpha,\alpha'\in[0,T]} \left(\frac{\sin((a/2)(\gamma-\gamma')\log T)}{(a/2)(\gamma-\gamma')\log T}\right) w(\gamma-\gamma'),$$

and hence

$$\sum_{\gamma\in[0,T]} m_\rho \le \sum_{\gamma,\gamma'\in[0,T]} \left(\frac{\sin((a/2)(\gamma-\gamma')\log T)}{(a/2)(\gamma-\gamma')\log T}\right)^2 w(\gamma-\gamma').$$

Setting $a=1-\delta$ for some small $\delta>0$ and applying Corollary 1, as $T\to\infty$

$$\sum_{\gamma\in[0,T]} m_\rho \le \left(\frac{1}{1-\delta}+\frac{1-\delta}{3}\right)\frac{T}{2\pi}\log T = \left(\frac{4}{3}+o(1)\right)\frac{-T}{2\pi}\log T.$$

On the other hand we have

$$\sum_{\substack{\gamma\in[0,T]\\ \rho \text{ simple}}} 1 \ge \sum_{\gamma\in[0,T]} (2-m_\rho).$$

Hence,

$$\sum_{\substack{\gamma\in[0,T]\\ \rho \text{ simple}}} 1 \ge 2N(T) - \sum_{\gamma\in[0,T]} m_\rho,$$

where $N(T)$ denotes the number of zeros of $\zeta(s)$ in the critical strip $0 \leq \sigma \leq 1$ with height less than equal to T. Since

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + o(\log T),$$

as $T \to \infty$

$$\sum_{\substack{\gamma \in [0,T] \\ \rho \text{ simple}}} 1 \geq 2\left(\frac{T}{2\pi} \log T\right) - \left(\frac{4}{3} + o(1)\right) \frac{T}{2\pi} \log T.$$

In the limit we have

$$\sum_{\substack{\gamma \in [0,T] \\ \rho \text{ simple}}} 1 \geq \left(\frac{2}{3} + o(1)\right) \frac{T}{2\pi} \log T.$$

□

Corollary 2 tells us that at least $\frac{2}{3}$ of the zeros of $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$ are simple. This result is proved using the RH. There are other results about the order of the zeros on the critical line $\sigma = \frac{1}{2}$ which do not assume RH. For example, Selberg showed that a positive density of the zeros of $\zeta(s)$ lie on the critical line and has odd order.

Montgomery also proved a third corollary in which he interprets Theorem 1 in terms of prime numbers, which we will state here without proof. Ordering the imaginary parts of zeros of $\zeta(s)$ in the upper half-plane as $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq, \ldots,$ the asymptotic formula for $N(T)$ shows that the average spacing $\gamma_{n+1} - \gamma_n$ between consecutive zeros $\frac{2\pi}{\log \gamma_n}$. Montgomery showed that the differences between consecutive zeros are not always near the average.

Corollary 3. *(Assume RH.) There exists a constant λ such that*

$$\lim_{n \to \infty} (\gamma_{n+1} - \gamma_n) \frac{\log \gamma_n}{2\pi} \leq \lambda \leq 1.$$

Chapter 3

Ledoan and Zaharescu's Method

In 2011 Ledoan and Zaharescu [4] , [5] considered the function H_λ defined by

$$H_\lambda = \zeta\left(s - \frac{i\lambda}{2}\right)\zeta\left(s + \frac{i\lambda}{2}\right),$$

where λ is a fixed positive real number. Figure 3.1 shows the first twenty zeros in the upper half-plane and their complex conjugates with respect to the real axis. Figure 3.2 shows a surface plot of $1/|H_\lambda(s)|$ showing its zeros on the critical line as divergences.

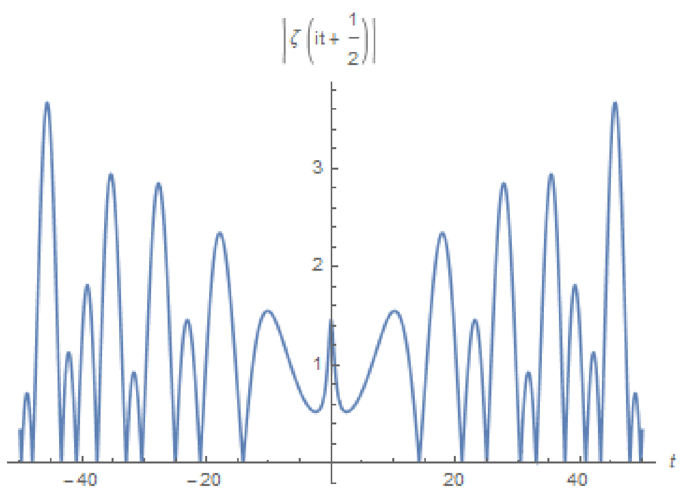

Figure 3.1: Zeros of $H_\lambda(s)$ on the critical line, with $t \in [-50, 50]$.

The purpose of the research of Ledoan and Zaharescu was to carry through deprivation and prove explicit formulas of H_λ. It is well known that explicit formulas were originally motivated by the counting of primes(see reference by A.E. Ingram [17].) We can summarize their first result as follows.

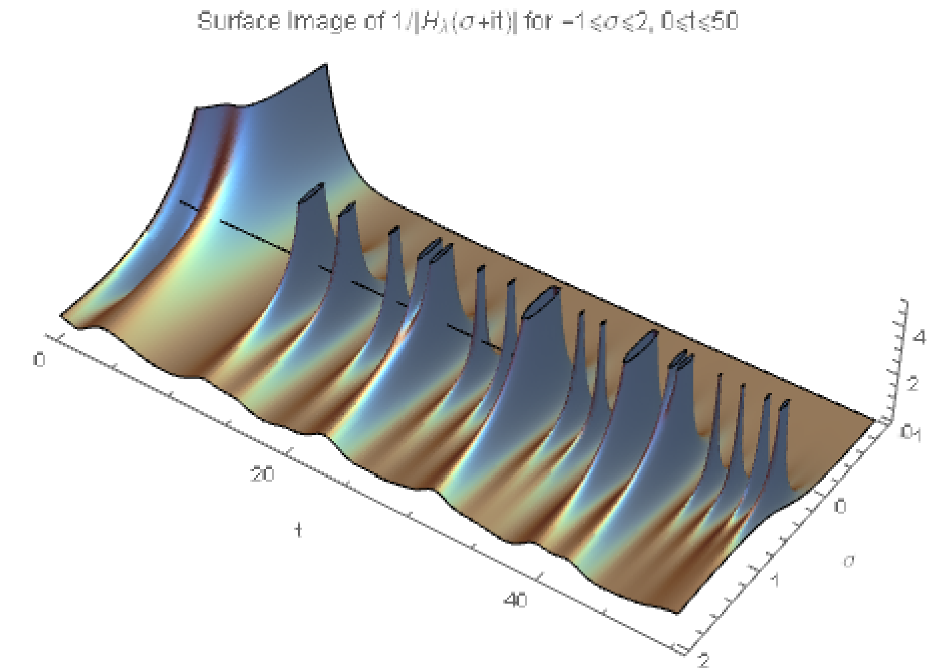

Figure 3.2: Surface plot of $H_\lambda(s)$ for $-1 \leq \sigma \leq 2$ and $0 \leq t \leq 50$.

Theorem 2. *Fix a positive real number λ for all $2 < x < T$ we have*

$$\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T \leq Im(\rho), Im(\rho') \leq T}} \frac{x^{\rho+\rho'}}{\rho+\rho'} = \frac{2xT}{\pi}\left\{\left[1 + Re\left(\frac{x^{i\lambda}}{1+i\lambda}\right)\right]\log x - Re\left(\frac{x^{i\lambda}}{(1+i\lambda)^2}\right)^{-1}\right\}$$

$$+ O_\lambda\left(xT\exp(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right) + O\left(x^2(\log T)^4\right)$$

$$+ O\left(T(\log T)^3\right),$$

where c is a positive absolute constant.

They obtained general formulas of the correlation of zeros of $H_\lambda(s)$ and proved the following theorem.

Theorem 3. *Fix a positive real number λ for all $T > 2$ and any continuously differentiable* *complex-valued function g with support contined in the interval* $(2, T)$*, we have,*

$$\sum_{\substack{H_\lambda(\rho)=0\\H_\lambda(\rho')=0\\-T\le Im(\rho),Im(\rho')\le T}} f(\rho+\rho') = \frac{2T}{x}\int_0^\infty g(x)(1+\cos(\lambda\log x))\log x dx$$

$$+ O_\lambda\left(T\|g'(x)x\exp(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\|_1\right)$$
$$+ O_\lambda\left((\log T)^4\|g'(x)x^2\|_1\right) + O_\lambda\left(T(\log T)^3\|g'(x)\|_1\right),$$

where f is a Mellin transform of g and c is a positive absolute constant.

The authors considered Montgomery's weight function $w(z)$ given by

$$w(z) = \frac{4}{4-z^2}$$

and defined for any real number α

$$\mathcal{F}_{H_\lambda}(\alpha) = \mathcal{F}_{H_\lambda}(\alpha, T) = \sum_{\substack{H_\lambda(\rho)=0\\H_\lambda(\rho')=0\\-T\le \mathrm{Im}(\rho),\mathrm{Im}(\rho')\le T}} T^{\alpha(\rho+\rho'-1)}w(\rho+\rho'-1).$$

They established the following asymptotic formula for $\mathcal{F}_{H_\lambda}$.

Theorem 4. *Fix n positive real number* λ *and* $\alpha \in (0, 1)$*. We have,*

$$\mathcal{F}_{H_\lambda}(\alpha) = 2\alpha\left(1+\frac{4\cos(\lambda\log T)}{4+\lambda^2}\right) - \frac{16x\sin(\alpha\log T)}{(4+\lambda^2)^2\log T}$$
$$- O_\lambda(\exp(-\alpha c(\log T)^{\frac{3}{5}}(\log\log T)^{-\frac{1}{5}})),$$

where c is a positive absolute constant.

3.1 Explicit Formula for the Pair Correlation of Zeros of $H_\lambda(s)$

To prove Theorem 2 we fix a positive real number λ and take $x \in (1, T]$. We write

$$\sum_{\substack{H_\lambda(\rho)=0\\-T\le \mathrm{Im}(\rho)\le T}} x^\rho = S_1 + S_2 + S_3 + S_4 + S_5 - S_6$$

where

$$S_1 = x^{i\frac{\lambda}{2}} \sum_{\substack{\zeta(\tilde{\rho}) \\ -T\le \mathrm{Im}(\tilde{\rho})\le T}} x^{\tilde{\rho}}, \quad S_2 = \sum_{\substack{\zeta(\tilde{\rho}) \\ -T-\frac{\lambda}{2}\le \mathrm{Im}(\tilde{\rho})\le -T}} x^{\tilde{\rho}},$$

$$S_3 = \sum_{\substack{\zeta(\tilde{\rho}) \\ T-\frac{\lambda}{2}\le \mathrm{Im}(\tilde{\rho})\le T}} x^{\tilde{\rho}}, \quad S_4 = x^{-i\frac{\lambda}{2}} \sum_{\substack{\zeta(\tilde{\rho}) \\ -T\le \mathrm{Im}(\tilde{\rho})\le T}} x^{\tilde{\rho}},$$

$$S_5 = x^{-i\frac{\lambda}{2}} \sum_{\substack{\zeta(\tilde{\rho}) \\ T\le \mathrm{Im}(\tilde{\rho})\le T+\frac{\lambda}{2}}} x^{\tilde{\rho}}, \quad S_6 = x^{-i\frac{\lambda}{2}} \sum_{\substack{\zeta(\tilde{\rho}) \\ -T\le \mathrm{Im}(\tilde{\rho})\le -T+\frac{\lambda}{2}}} x^{\tilde{\rho}}.$$

From Chapter 15 in the classical book by Davenport [3], we know that the number of zeros of $\zeta(s)$ with imaginary parts in the interval $[-T-\frac{\lambda}{2}, -T+\frac{\lambda}{2}] \cup [T-\frac{\lambda}{2}, T+\frac{\lambda}{2}]$ is at most $O_\lambda(\log T)$. Furthermore, $|x^{\tilde{\rho}}| < x$ for all $\tilde{\rho}$. For these reasons each of the sums S_2, S_3, S_5, and S_6 is at most $O_\lambda(x \log T)$. So, we must examine the remaining sums S_1 and S_4. To do this we shall use the Landau-Gonek asymptotic formula written in the form,

$$\sum_{\substack{\zeta(\rho) \\ -T-\frac{\lambda}{2}\le \mathrm{Im}(\rho)\le -T}} x^{\rho} = \frac{-\Lambda(n_x)}{\pi} \cdot \frac{\sin(T\log x/n_x)}{\log x/n_x} + O\left(x(\log 2xT)^2 + \frac{\log 2T}{\log x}\right),$$

where $x, T > 1$, n_x is the nearest prime power to x, and Λ is the Mongoldt function. From this asymptotic formula, we can derive an equivalent formula for $H_\lambda(x)$,

$$\sum_{\substack{\zeta(\rho) \\ -T-\frac{\lambda}{2}\le \mathrm{Im}(\rho)\le -T}} x^{\rho} = \frac{-\Lambda(n_x)}{\pi} \cdot \frac{\sin(T\log x/n_x)}{\log x/n_x}\left(x^{\frac{i\Lambda}{2}} + x^{\frac{-i\Lambda}{2}}\right) + O\left(x(\log 2xT)^2 + \frac{\log 2T}{\log x}\right),$$

Squaring both sides of this relation and multiplying the result by x^{-1},

$$\begin{aligned}\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le \mathrm{Im}(\rho),\mathrm{Im}(\rho')\le -T}} x^{\rho+\rho'-1} &= \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2\left(x^{-1+i\lambda} + x^{-1-i\lambda} + 2x^{-1}\right) \\ &+ O\left(x(\log 2xT)^4 + \left(\frac{\log 2T}{\log x}\right)\right) \\ &+ O_\lambda\left(\log x\left((\log 2xT)^2 + \frac{\log 2T}{\log x}\right)\left|\frac{\sin(T\log x/n_x)}{\log x/n_x}\right|\right).\end{aligned}$$

Integrating this relation with respect to y from 2 to y for some $y \in [2, T]$,

$$\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\leq \mathrm{Im}(\rho),\mathrm{Im}(\rho')\leq -T}} \frac{y^{\rho+\rho'} - 2^{\rho+\rho'}}{\rho+\rho'} = \int_2^y \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2 x^{-1+i\lambda}dx$$

$$+ \int_2^y \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2 x^{-1-i\lambda}dx$$
$$+ \int_2^y \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2 2x^{-1}dx$$
$$+ O_\lambda\left(\int_2^y \log\left((\log 2xT)^2 + \frac{\log 2T}{\log x}\right)\left|\frac{\sin(T\log x/n_x)}{\log x/n_x}\right|\right)dx.$$

For the sake of brevity, let

$$G(u) = \int_2^y \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2 x^{-1+iu}dx.$$

This lets us to write

$$\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\leq \mathrm{Im}(\rho),\mathrm{Im}(\rho')\leq -T}} \frac{y^{\rho+\rho'} - 2^{\rho+\rho'}}{\rho+\rho'}$$
$$= G(\lambda) + G(-\lambda) + G(0) + O_\lambda\left(\int_2^y \log\left((\log 2xT)^2 + \frac{\log 2T}{\log x}\right)\left|\frac{\sin(T\log x/n_x)}{\log x/n_x}\right|\right)dx.$$

We have $n_x = q_{k+1}$ for each $x \in (x_k, x_{k+1})$, where $k > 0$, so that $\Lambda(n_x) = \Lambda(q_{k+1})$. Letting $x_l < x < y < x_{l+1}$, $l > 0$, we write

$$G(u) = \left(\int_2^{x_1} + \int_{x_1}^{x_2} + \ldots + \int_{x_{l-1}}^{x_l} + \int_{x_l}^{y}\right)\frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2 x^{-1+iu}dx.$$

Let $t = T\log x/q_m$. We have

$$\int_{x_{m-1}}^{x_m} \frac{\Lambda(n_x)^2}{\pi^2}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2 x^{-1+iu}dx = \frac{\Lambda(q_m)^2}{\pi^2}\int_{\frac{q_{m-1}+q_m}{2}}^{\frac{q_m+q_{m+1}}{2}}\left(\frac{\sin(T\log x/n_x)}{\log x/n_x}\right)^2 x^{-1+iu}dx$$
$$= \frac{\Lambda(q_m)^2 q_m^{iu} T}{\pi^2}\int_{\frac{T\log q_{m-1}+q_m}{2q_m}}^{\frac{T\log q_m+q_{m+1}}{2q_m}} e^{\frac{iut}{T}}\left(\frac{\sin t}{t}\right)^2 dt,$$

where

$$\int_{\frac{T\log q_{m-1}+q_m}{2q_m}}^{\frac{T\log q_m+q_{m+1}}{2q_m}} e^{\frac{iut}{T}}\left(\frac{\sin t}{t}\right)^2 dt = \int_{-\infty}^{\infty} e^{\frac{iut}{T}}\left(\frac{\sin t}{t}\right)^2 dt - E_1 - E_2,$$

where

$$E_1 = \int_{-\infty}^{\frac{T \log q_{m-1}+q_m}{2q_m}} e^{\frac{iut}{T}} \left(\frac{\sin t}{t}\right)^2 dt \ll_u \int_{-\infty}^{\frac{T \log q_{m-1}+q_m}{2q_m}} \frac{1}{t^2} dt$$

and

$$E_2 = \int_{\frac{T \log q_m+q_{m+1}}{2q_m}}^{\infty} e^{\frac{iut}{T}} \left(\frac{\sin t}{t}\right)^2 dt \ll_u \int_{\frac{T \log q_m+q_{m+1}}{2q_m}}^{\infty} \frac{1}{t^2} dt$$

Using the Taylor series expansion with $|t| < \sqrt{T}$,

$$\int_{-\infty}^{\infty} e^{\frac{iut}{T}} \left(\frac{\sin t}{t}\right)^2 dt = \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt + O_u\left(\frac{1}{\sqrt{T}}\right) = \pi + O_u\left(\frac{1}{\sqrt{T}}\right).$$

To bound the error E_2, we note that

$$T \log\left(\frac{q_m + q_{m+1}}{2q_m}\right) \geq T \log\left(\frac{1+2q_m}{2q_m}\right) \gg \frac{T}{2q_m} \gg \frac{T}{q},$$

and hence

$$E_2 \ll_u \int_{\frac{T}{y}}^{\infty} \frac{1}{t^2} dt \ll_u \frac{y}{T}.$$

By similar reasoning we find that

$$E_1 \ll_u \frac{y}{T}.$$

Altogether,

$$\int_{\frac{T \log q_{m-1}+q_m}{2q_m}}^{\frac{T \log q_m+q_{m+1}}{2q_m}} e^{\frac{iut}{T}} \left(\frac{\sin t}{t}\right)^2 dt = \pi + O_u\left(\frac{1}{\sqrt{T}}\right) + O_u\left(\frac{y}{T}\right).$$

As a result we have

$$\int_{x_{m-1}}^{x_m} \frac{\Lambda(n_x)^2}{\pi^2} \left(\frac{\sin\left(T \log \frac{x}{n_x}\right)}{\log \frac{x}{n_x}}\right)^2 x^{-1+iu} dx = \frac{\Lambda(q_m)^2 q_m^{iu} T}{\pi^2}\left(\pi + O_u\left(\frac{1}{\sqrt{T}}\right) + O_u\left(\frac{y}{T}\right)\right)$$

The above calculations give us

$$\begin{aligned}
G(u) = & \sum_{\substack{q \leq y \\ q \text{ prime power}}} \frac{\Lambda(n_x)^2}{\pi^2} \frac{\Lambda(q_m)^2 q_m^{iu} T}{\pi^2}\left(\pi + O_u\left(\frac{1}{\sqrt{T}}\right) + O_u\left(\frac{y}{T}\right)\right) \\
&= \frac{T}{\pi} \sum_{\substack{q \leq y \\ q \text{ prime power}}} \Lambda(q)^2 q^{iu} + O_u\left(\sqrt{T} \sum_{\substack{q \leq y \\ q \text{ prime power}}} \Lambda(q)^2\right) + O_u\left(y \sum_{\substack{q \leq y \\ q \text{ prime power}}} \Lambda(q)^2\right) \\
&= \frac{T}{\pi} \sum_{\substack{q \leq y \\ q \text{ prime power}}} \Lambda(q)^2 q^{iu} + O_u\left(y^2 \log y\right),
\end{aligned}$$

using

$$\sum_{\substack{q\leq y\\ q \text{ prime power}}} \Lambda(q)^2 \leq \log y \sum_{\substack{q\leq y\\ q \text{ prime power}}} \Lambda(q) \sim y\log y,$$

as y tends to infinity.

Here, we note that

$$\sum_{\substack{q\leq y\\ q \text{ prime power}}} \Lambda(q)^2 q^{iu} = \sum_{\substack{q\leq y\\ q \text{ prime power}}} \Lambda(q) q^{iu}\log y + \sum_{\substack{q\leq y\\ \text{prime power}}} \Lambda(q)\left(\Lambda(q)-\log y\right) q^{iu}.$$

By the prime number theorem,

$$\pi(\sqrt{y}) + \pi(\sqrt[3]{y}) + \ldots \ll \frac{\sqrt{y}}{\log y}.$$

Furthermore, since $\Lambda(q) = \log q$ for each prime q,

$$\left|\sum_{\substack{q\leq y\\ q \text{ prime power}}} \Lambda(q)\left(\Lambda(q)-\log y\right)\right| \ll \sqrt{y}\log y.$$

Next, applying the summation by parts formula to

$$\sum_{\substack{q\leq y\\ q \text{ prime power}}} \Lambda(q) q^{iu}\log y = \sum_{n\leq y} \Lambda(n) n^{iu}\log n$$

to obtain the following

$$\sum_{2<n\leq y} \Lambda(n) n^{iu}\log n = \Psi(y) y^{iu}\log y - \Psi(2)2^{iu}\log 2 - \int_2^y \Psi(t) t^{-1+iu}(1+iu\log t)dt \tag{3.1}$$

where

$$Psi(x) = \sum_{p^\rho\leq x}\log p = \sum_{n\leq x}\Lambda(n),$$

with $x>0$, p is a prime, and ν is a positive integer. Using the zero-free region for $\zeta(s)$ due to Korobov and Vinogradov (see Titchmarch's classical book [8]),

$$\Psi(x) = x + O(x\exp(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}})), \tag{3.2}$$

where c is a positive absolute constant, we obtain

$$\begin{aligned}\int_2^y \Psi(t) t^{-1+iu}(1+iu\log t)dt = &\int_2^y t^{iu}(1+iu\log t)dt \\ &+ O_u\left(\int_2^y \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{\frac{1}{5}}\log t dt)\right).\end{aligned}$$

Concerning the integral on the right-hand side in the above equation we see that

$$\int_2^y \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{\frac{1}{5}} \log t dt = \int_2^{\sqrt{y}} \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{\frac{1}{5}} \log t dt$$
$$+ \int_{\sqrt{y}}^y \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{-\frac{1}{5}} \log t dt$$

Here, we note that

$$\int_2^{\sqrt{y}} \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{\frac{1}{5}} \log t dt \ll \int_2^{\sqrt{y}} dt \ll \sqrt{y}$$

and

$$\int_{\sqrt{y}}^y \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{\frac{1}{5}} \log t dt \leq \int_{\sqrt{y}}^y \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{\frac{1}{5}} \log \sqrt{y} dt$$
$$\ll y \exp(-c'(\log y)^{\frac{3}{5}}(\log\log y)^{-\frac{1}{5}})$$

where c' is a positive absolute constant. Altogether,

$$\int_2^y \Psi(t)t^{-1+iu}(1+iu\log t)dt = \int_2^y t^{iu}(1+iu\log t)dt$$
$$+ O_u\left(y\int_2^y \exp(-c(\log t)^{\frac{3}{5}}(\log\log t)^{\frac{1}{5}}) \log t dt\right),$$

Integrating by parts,

$$\int_2^y t^{iu}(1+iu\log t)dt = \frac{y^{1+iu}}{1+iu}\left(1+iu\log y - \frac{iu}{1+iu}\right) - \frac{2^{1+iu}}{1+iu}\left(1+iu\log 2 - \frac{iu}{1+iu}\right),$$

and hence

$$\int_2^y \Psi(t)t^{-1+iu}(1+iu\log t)dt = \frac{y^{1+iu}}{1+iu}\left(1+iu\log y - \frac{iu}{1+iu}\right)$$
$$+ O_u(y\exp(-c'(\log y)^{\frac{3}{5}}(\log\log y)^{-\frac{1}{5}}).$$

Inserting this into (3.1) and using the prime number theorem in (3.2),

$$\sum_{n\leq x} \Lambda(n)n^{iu}\log n = y^{1+iu}\left(1-\frac{iu}{1+iu}\right)\left(\log y - \frac{1}{1+iu}\right)$$
$$+ O_u(y\exp(-c\prime(\log y)^{\frac{3}{5}}(\log\log y)^{-\frac{1}{5}})$$

As a result,

$$\sum_{\substack{q\leq y \\ q\text{prime power}}} \Lambda(q)^2 = y^{1+iu}\left(1-\frac{iu}{1+iu}\right)\left(\log y - \frac{1}{1+iu}\right)$$
$$+ O_u(y\exp(-c\prime(\log y)^{\frac{3}{5}}(\log\log y)^{-\frac{1}{5}}.$$

From this we find that

$$G(u) = \frac{y^{1+iu}T}{\pi}\left(1 - \frac{iu}{1+iu}\right)\left(\log y - \frac{1}{1+iu}\right) + O_u(yT\exp(-c'(\log y)^{\frac{3}{5}}(\log\log y)^{-\frac{1}{5}}$$
$$+ O(y^2 \log y)$$

Using this asympototic formula with $u = \lambda$, $u = -\lambda$, $u = 0$ and putting the result into (2.26), we obtain

$$\begin{aligned}\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\le \mathrm{Im}(\rho),\mathrm{Im}(\rho')\le T}} \frac{y^{\rho+\rho'}}{\rho+\rho'} &= \frac{2yT}{\pi}\left[\left(1+\mathrm{Re}\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right)\log y - \mathrm{Re}\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right)^{-1}\right] \\ &\quad + O_\lambda\left(yT\exp(-c'(\log y)^{\frac{3}{5}}(\log\log y)^{-\frac{1}{5}}\right) \\ &\quad + O(y^2\log y) + O\left(y^2(\log T)^4\right) \\ &\quad + O_\lambda\left(\int_2^y \log x\left((\log 2xT)^2 + \frac{\log 2T}{x\log x}\right)\left|\frac{\sin\left(T\log\frac{x}{n_x}\right)}{\log\frac{x}{n_x}}\right| dx\right). \end{aligned} \tag{3.3}$$

Next, we have

$$\int_2^y \log x\left((\log 2xT)^2 + \frac{\log 2T}{x\log x}\right)\left|\frac{\sin\left(T\log\frac{x}{n_x}\right)}{\log\frac{x}{n_x}}\right| dx \le (\log y)(\log 2yT)^2\int_2^y \left|\frac{\sin\left(T\log\frac{x}{n_x}\right)}{\log\frac{x}{n_x}}\right| dx.$$

To estimate the integral on the right-hand side, we set $t = T\log\frac{x}{q_m}$ and note that for each $2 \le m \le l$,

$$\begin{aligned}\int_{x_{m-1}}^{x_m}\left|\frac{\sin(T\log x/n_x)}{\log x/n_x}\right| dx &= \int_{x_{m-1}}^{x_m}\left|\frac{\sin(T\log x/q_m)}{\log x/q_m}\right| dx \\ &= q_m\int_{\frac{T\log q_{m-1}+q_m}{2q_m}}^{\frac{T\log q_m+q_{m+1}}{2q_m}} e^{\frac{t}{T}}\left|\frac{\sin t}{t}\right| dt \\ &< q_m\int_{-T\log\frac{4}{3}}^{T\log\frac{3}{2}} e^{\frac{t}{T}}\left|\frac{\sin t}{t}\right| dt \\ &\le \frac{3q_m}{2}\int_{-T\log\frac{4}{3}}^{T\log\frac{3}{2}}\left|\frac{\sin t}{t}\right| dt \\ &\ll \log T,\end{aligned}$$

since $q_{m+1} < 2q_m$, $q_{m-1} > \frac{q_m}{2}$, and since

$$T\log\left(\frac{q_m+q_{m+1}}{2q_m}\right) \le T\log\frac{3}{2}$$

and

$$T \log\left(\frac{q_{m-1}+q_m}{2q_m}\right) \geq -T\log\frac{4}{3}$$

if $t \leq T\log\frac{3}{2}$. Hence,

$$\int_2^y \left|\frac{\sin(T\log x/n_x)}{\log x/n_x}\right| dx \ll \sum_{\substack{q\leq y\\ q \text{ prime power}}} q\log T \ll \frac{y^2\log T}{\log y}.$$

Combining all estimates,

$$\int_2^y \log x\left((\log 2xT)^2 + \frac{\log 2T}{x\log x}\right)\left|\frac{\sin(T\log x/n_x)}{\log x/n_x}\right| dx = O\left(y^2(\log 2yT)^2\log T\right). \tag{3.4}$$

Inserting (3.4) into (3.3) and noting that $y\in[2,T]$,

$$\begin{aligned}\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\leq \mathrm{Im}(\rho),\mathrm{Im}(\rho')\leq T}} \frac{y^{\rho+\rho'}}{\rho+\rho'} &= \sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\leq \mathrm{Im}(\rho),\mathrm{Im}(\rho')\leq T}} \frac{y^{\rho+\rho'}}{\rho+\rho'} - \sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\leq \mathrm{Im}(\rho),\mathrm{Im}(\rho')\leq T}} \frac{2^{\rho+\rho'}}{\rho+\rho'}\\ &= \frac{2yT}{\pi}\left\{\left[1+\mathrm{Re}\left(\frac{y^{i\lambda}}{1+i\lambda}\right)\right]\log y - \mathrm{Re}\left(\frac{y^{i\lambda}}{(1+i\lambda)^2}\right)^{-1}\right\}\\ &\quad + O_\lambda(yT\exp(-c'(\log y)^{\frac{3}{5}}(\log\log y)^{-\frac{1}{5}}) + O(y^2(\log T)^4).\end{aligned} \tag{3.5}$$

Here, we observe that

$$\begin{aligned}\left|\sum_{\substack{H_\lambda(\rho)=0\\ H_\lambda(\rho')=0\\ -T\leq \mathrm{Im}(\rho),\mathrm{Im}(\rho')\leq T}} \frac{2^{\rho+\rho'}}{\rho+\rho'}\right| &\leq 4\sum_{\substack{H_\lambda(\rho)=0\\ -T\leq\mathrm{Im}(\rho)\leq T}}\sum_{0\leq k<2T}\sum_{\substack{H_\lambda(\rho')=0\\ -T\leq \mathrm{Im}(\rho')\leq T\\ |\mathrm{Im}(\rho+\rho')|\in[k,k+1]}} \frac{1}{\rho+\rho'}\\ &\leq 4\sum_{\substack{H_\lambda(\rho)=0\\ -T\leq\mathrm{Im}(\rho)\leq T}}\sum_{\substack{H_\lambda(\rho')=0\\ -T\leq \mathrm{Im}(\rho')\leq T\\ |\mathrm{Im}(\rho+\rho')|\in[0,1]}} \frac{1}{|\rho+\rho'|}\\ &\quad + 4\sum_{\substack{H_\lambda(\rho)=0\\ -T\leq\mathrm{Im}(\rho)\leq T}}\sum_{1\leq k<2T}\sum_{\substack{H_\lambda(\rho')=0\\ -T\leq \mathrm{Im}(\rho')\leq T\\ |\mathrm{Im}(\rho+\rho')|\in[0,1]}} \frac{1}{|\rho+\rho'|}.\end{aligned}$$

Since $\mathrm{Re}(\rho), \mathrm{Im}(\rho') \gg \frac{1}{\log T}$, we have

$$\begin{aligned}
4 \sum_{\substack{H_\lambda(\rho)=0 \\ -T\le \mathrm{Im}(\rho)\le T}} \sum_{\substack{H_\lambda(\rho')=0 \\ -T\le \mathrm{Im}(\rho')\le T \\ |\mathrm{Im}(\rho+\rho')|\in[0,1]}} \frac{1}{|\rho+\rho'|} &\le 4 \sum_{\substack{H_\lambda(\rho)=0 \\ -T\le \mathrm{Im}(\rho)\le T}} \sum_{\substack{H_\lambda(\rho')=0 \\ -T\le \mathrm{Im}(\rho')\le T \\ |\mathrm{Im}(\rho+\rho')|\in[0,1]}} \frac{1}{\mathrm{Re}(\rho+\rho')} \\
&\ll \sum_{\substack{H_\lambda(\rho)=0 \\ -T\le \mathrm{Im}(\rho)\le T}} \sum_{\substack{H_\lambda(\rho')=0 \\ -T\le \mathrm{Im}(\rho')\le T \\ |\mathrm{Im}(\rho+\rho')|\in[0,1]}} \log T \\
&\ll (\log T)^2 |\{\rho\colon H_\lambda(\rho)=0, -T\le \mathrm{Im}(\rho)\le T\}| \\
&\ll T(\log T)^3.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
4 \sum_{\substack{H_\lambda(\rho)=0 \\ -T\le \mathrm{Im}(\rho)\le T}} \sum_{1\le k<2T} \sum_{\substack{H_\lambda(\rho')=0 \\ -T\le \mathrm{Im}(\rho')\le T \\ |\mathrm{Im}(\rho+\rho')|\in[0,1]}} \frac{1}{|\rho+\rho'|} &\ll \sum_{\substack{H_\lambda(\rho)=0 \\ -T\le \mathrm{Im}(\rho)\le T}} \sum_{1\le k<2T} \frac{\log T}{k} \\
&\ll (\log T)^2 \sum_{\substack{H_\lambda(\rho)=0 \\ -T\le \mathrm{Im}(\rho)\le T}} 1 \\
&\ll T(\log T)^3.
\end{aligned}$$

Hence,

$$\left| \sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le \mathrm{Im}(\rho),\mathrm{Im}(\rho')\le T}} \frac{2^{\rho+\rho'}}{\rho+\rho;} \right| \ll T(\log T)^3. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}
\sum_{\substack{H_\lambda(\rho)=0 \\ H_\lambda(\rho')=0 \\ -T\le \mathrm{Im}(\rho),\mathrm{Im}(\rho')\le T}} \frac{x^{\rho+\rho'}}{\rho+\rho'} = \frac{2xT}{\pi} &\left\{ \left[1 + \mathrm{Re}\left(\frac{x^{i\lambda}}{1+i\lambda}\right)\right] \log x - \mathrm{Re}\left(\frac{x^{i\lambda}}{(1+i\lambda)^2}\right)^{-1} \right\} \\
&+ O_\lambda\left(xT\exp(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right) + O\left(x^2(\log T)^4\right) \\
&+ O\left(T(\log T)^3\right).
\end{aligned}$$

This concludes the proof of Theorem 2.

3.2 Some Consequences

We worked hard to obtain an asymptotic formula for the function $\mathcal{F}_{H_\lambda}(\alpha)$ in the form

$$\mathcal{F}_{H_\lambda}(\alpha) = (\log T + o(\log T))T^{-2\alpha} + \alpha + o(1), \quad T \to \infty,$$

where $\lambda > 0$ and $\delta > 0$, and to show that this formula holds uniformly for all $T \geq 2$ and all $\alpha \in [(3 \log\log T) \log T, 1-\delta]$. However, we were not successful.

Instead, we proved a consequence of Theorem 2, which may be used to investigate the simple zeros of $H_\lambda(s)$, and hence the simple zeros of $\zeta(s)$. Specifically, Theorem 5 (below) is an analogue of Montgomery's Corollary 1 which is applicable to the function $H_\lambda(s)$.

Theorem 5. *(Assume RH.) For any fixed real number λ, any fixed $\delta > 0$, and any fixed $0 < \alpha < 1-\delta$, we have*

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ H_\lambda(1/2+i\gamma)=0 \\ H_\lambda(1/2+i\gamma')=0}} \left(\frac{\sin(\alpha/2)(\gamma-\gamma')\log T}{(\alpha/2)(\gamma-\gamma')\log T}\right)^2 w(\gamma-\gamma') = \frac{T\log T}{\pi}\left(\frac{\alpha}{3}\right) + O\left(\frac{1}{\log T}\right).$$

To commence, we let

$$H(\beta) = \frac{\alpha - |\beta|}{\alpha^2}.$$

We compute that

$$\begin{aligned}
\int_{-\infty}^{\infty} H(\beta)\cos\beta x\, d\beta &= 2\int_0^{\alpha} H(\beta)\cos\beta x\, d\beta \\
&= \frac{2}{\alpha^2}\int_0^{\alpha}(\alpha-\beta)\cos\beta x\, d\beta \\
&= \frac{2}{\alpha^2}\int_0^{\alpha}(\alpha-\beta)\left(\frac{\sin\beta x}{x}\right)' d\beta \\
&= \frac{2}{\alpha^2}(\alpha-\beta)\left(\frac{\sin\beta x}{x}\right)\Bigg|_0^{\alpha} + \frac{2}{\alpha^2}\int_0^{\alpha}\frac{\sin\beta x}{x}\, d\beta \\
&= \frac{2}{\alpha^2 x}\int_0^{\alpha}\sin\beta x\, d\beta \\
&= -\frac{2}{\alpha^2x^2}\cos\beta x\Bigg|_0^{\alpha} \\
&= \frac{2}{\alpha^2x^2}(1-\cos\alpha x) \\
&= \frac{4}{\alpha^2x^2}\sin^2\frac{\alpha x}{2} \\
&= \left(\frac{\sin(\alpha x/2)}{\alpha x/2}\right)^2,
\end{aligned}$$

using integration by parts. Here, we point out that this holds for all $x \neq 0$. Hence,

$$\int_{-\infty}^{\infty} H(\beta) \cos \beta y \, d\beta = \left(\frac{\sin(\alpha y/2)}{\alpha y/2} \right)^2$$

for all $y \neq 0$. Let

$$x = \frac{\alpha y}{2},$$

so that

$$y = \frac{2x}{\alpha}.$$

As a result,

$$\int_{-\alpha}^{\alpha} H(\beta) \cos \frac{2\beta x}{\alpha} \, d\beta = \left(\frac{\sin x}{x} \right)^2$$

for all $x \neq 0$. Next, putting

$$x = \frac{\alpha(\gamma - \gamma') \log T}{2},$$

we obtain

$$\begin{aligned}
&\sum_{\substack{0<\gamma,\gamma' \leq T \\ H_\lambda(1/2+i\gamma)=0 \\ H_\lambda(1/2+i\gamma')=0}} \left(\frac{\sin(\alpha(\gamma - \gamma') \log T/2)}{\alpha(\gamma - \gamma') \log T/2} \right) w(\gamma - \gamma') \\
&\qquad = \sum_{\substack{0<\gamma,\gamma' \leq T \\ H_\lambda(1/2+i\gamma)=0 \\ H_\lambda(1/2+i\gamma')=0}} w(\gamma - \gamma') \int_{-\alpha}^{\alpha} H(\beta) \cos \left(\frac{2\beta}{\alpha} \cdot \frac{\alpha(\gamma - \gamma') \log T}{2} \right) d\beta \\
&\qquad = \sum_{\substack{0<\gamma,\gamma' \leq T \\ H_\lambda(1/2+i\gamma)=0 \\ H_\lambda(1/2+i\gamma')=0}} w(\gamma - \gamma') \int_{-\alpha}^{\alpha} H(\beta) \cos(\beta(\gamma - \gamma') \log T) \, d\beta.
\end{aligned}$$

Since

$$\int_{-\alpha}^{\alpha} H(\beta) \sin \frac{2\beta x}{\alpha} \, d\beta = 0,$$

we have

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} \left(\frac{\sin(\alpha(\gamma-\gamma')\log T/2)}{\alpha(\gamma-\gamma')\log T/2}\right) w(\gamma-\gamma')$$

$$= \sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} w(\gamma-\gamma')\int_{-\alpha}^{\alpha} H(\beta)(\cos(\beta(\gamma-\gamma')\log T)+\sin(\beta(\gamma-\gamma')\log T))\,d\beta$$

$$= \sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} w(\gamma-\gamma')\int_{-\alpha}^{\alpha} H(\beta)e^{i\beta(\gamma-\gamma')\log T}\,d\beta$$

$$= \sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} w(\gamma-\gamma')\int_{-\alpha}^{\alpha} H(\beta)T^{i\beta(\gamma-\gamma')}\,d\beta$$

$$= \int_{-\alpha}^{\alpha} H(\beta)\left(\sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} T^{i\beta(\gamma-\gamma')}w(\gamma-\gamma')\right)d\beta.$$

By a short calculation involving the sum

$$S_\lambda(T,x) = \sum_{\substack{0<\gamma\leq T\\ H_\lambda(1/2+i\gamma)=0}} x^{i\gamma} = (1+x^{i\lambda})\sum_{\substack{0<\gamma\leq T\\ \zeta(1/2+i\gamma)=0}} x^{i\gamma} + O_\lambda(\log T)$$

(counting multiplicity) and making use of

$$\int_2^y |S_\lambda(T,x)|^2\,dx = \int_2^y \sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} x^{i(\gamma-\gamma')}\,dx$$

$$= \sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} \frac{y^{1+i(\gamma-\gamma')}-2^{1+i(\gamma-\gamma')}}{1+i(\gamma-\gamma')}$$

$$= \sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} \frac{y^{1+i(\gamma-\gamma')}}{1+i(\gamma-\gamma')} + \text{small error}$$

$$= \frac{1}{4\pi}\sum_{2\leq m\leq y}(2+m^{i\lambda}+m^{-i\lambda})\Lambda^2(m) + \text{small error},$$

we find that

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} x^{1+i(\gamma-\gamma')}w(\gamma-\gamma') = \frac{T\log x}{2\pi}\left(\frac{x^{i\lambda}+x^{-i\lambda}}{4+\lambda^2}+\frac{1}{2}\right)+O_\lambda(T).$$

From this, we take $x = T^\alpha$ with $0 \leq \alpha \leq 1-\delta$, we deduced that

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma') = \frac{T\log T}{2\pi}\alpha\left(2+\frac{4}{4+\lambda^2}(T^{i\lambda\alpha}+T^{-i\lambda\alpha})\right)+O(T).$$

Using this with α replaced by β, we find that

$$\begin{aligned}
&\sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} \left(\frac{\sin(\alpha(\gamma-\gamma')\log T/2)}{\alpha(\gamma-\gamma')\log T/2}\right)w(\gamma-\gamma')\\
&\qquad = \frac{T\log T}{2\pi}\int_{-\alpha}^{\alpha} H(\beta)|\beta|\left(2+\frac{4}{4+\lambda^2}(T^{i\lambda\beta}+T^{-i\lambda\beta})+O(T)\right)d\beta\\
&\qquad = \frac{T\log T}{\pi}\int_{0}^{\alpha}\frac{(\alpha-\beta)\beta}{\alpha^2}\left(2+\frac{4}{4+\lambda^2}(T^{i\lambda\beta}+T^{-i\lambda\beta})\right)d\beta+O\left(\frac{1}{\log T}\right)\\
&\qquad = \frac{T\log T}{\pi}\int_{0}^{\alpha}\frac{(\alpha-\beta)\beta}{\alpha^2}\left(2+\frac{8}{4+\lambda^2}\cos(\lambda\beta\log T)\right)d\beta+O\left(\frac{1}{\log T}\right)\\
&\qquad = \frac{T\log T}{\pi}\cdot\frac{2}{\alpha^2}\int_{0}^{\alpha}(\beta\alpha-\beta^2)\,d\beta+O\left(\frac{1}{\log T}\right)\\
&\qquad = \frac{T\log T}{\pi}\cdot\frac{2}{\alpha^2}\left(\frac{\beta^2\alpha}{2}-\frac{\beta^3}{3}\right)\Bigg|_0^\alpha+O\left(\frac{1}{\log T}\right)\\
&\qquad = \frac{T\log T}{\pi}\cdot\frac{2}{\alpha^2}\left(\frac{\alpha^3}{2}-\frac{\alpha^3}{3}\right)+O\left(\frac{1}{\log T}\right)\\
&\qquad = \frac{T\log T}{\pi}\left(\frac{\alpha}{3}\right)+O\left(\frac{1}{\log T}\right),
\end{aligned}$$

and hence

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} \left(\frac{\sin(\alpha(\gamma-\gamma')\log T/2)}{\alpha(\gamma-\gamma')\log T/2}\right)w(\gamma-\gamma') = \frac{T\log T}{\pi}\left(\frac{\alpha}{3}\right)+O\left(\frac{1}{\log T}\right),$$

as required. This finishes the proof of Theorem 5.

We point out that, when $\lambda = 0$, the left-hand side gives multiplicity two for each zero. Hence, for pairs of zeros the multiplicity is four. The right-hand side gives four times

Montgomery's result. Finally, from the above calculations, we also have

$$\left(\frac{T\log T}{2\pi}\right)^{-1} \sum_{\substack{0\le\gamma,\gamma'\le T\\ H_\lambda(1/2+i\gamma)=0\\ H_\lambda(1/2+i\gamma')=0}} T^{\beta i(\gamma-\gamma')}w(\gamma-\gamma') = |\beta|\left(2+\frac{4}{4+\lambda^2}(T^{i\lambda\beta}+T^{-i\lambda\beta})\right) + O\left(\frac{1}{\log T}\right).$$

The left-hand side is a function of λ, β and T. It is easy to see that this function is an even function of β

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